COMPLETE 2-LOOP QUANTUM ELECTRODYNAMIC CONTRIBUTIONS TO THE MUON LIFETIME IN THE FERMI MODEL

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The complete 2-loop QED contributions to the muon lifetime have been calculated analytically in the Fermi theory. The exact result for the effects of virtual and real photons, virtual electrons, muons and hadrons as well as e^+e^- pair creation is

$$\Delta\Gamma^{(2)} = \Gamma_0 \left(\frac{\alpha}{\pi}\right)^2 \left(\frac{156815}{5184} - \frac{1036}{27}\zeta(2) - \frac{895}{36}\zeta(3) + \frac{67}{8}\zeta(4) + 53\zeta(2)\ln 2 - (0.042 \pm 0.002)\right)$$

where Γ_0 is the tree-level width. This eliminates the theoretical error in the extracted value of the Fermi coupling constant, G_F, which was previously the source of the dominant uncertainty. The new value is

$$G_F = (1.16637 \pm 0.00001) \times 10^{-5}\,\mathrm{GeV}^{-2}$$

with the error being entirely experimental. Several experiments are planned for the next generation of muon lifetime measurements and these can proceed unhindered by theoretical uncertainties.

1 Introduction

The three fundamental input parameters that enter into all calculations of electroweak physics are the electromagnetic coupling constant, α, the Fermi coupling constant, G_F, and the mass of the Z^0 boson, M_Z. Their current best values, along with their absolute and relative errors are [1,2]

$$\alpha = 1/(137.0359895 \pm 0.0000061) \qquad (0.045\,\mathrm{ppm})$$

$$G_F = (1.16639 \pm 0.00002) \times 10^{-5}\,\mathrm{GeV}^{-2} \qquad (17\,\mathrm{ppm})$$

$$M_Z = 91.1867 \pm 0.0021\,\mathrm{GeV} \qquad (23\,\mathrm{ppm})$$

The first of these, α, comes with the liability the 'hadronic uncertainty' arising because, when used for the analysis of data near the Z^0 resonance, it must be run up from a scale $q^2 = 0$ to M_Z^2 crossing, on the way, the hadronic resonance region.

In the mid-80's, just before the turn on of LEP, a CERN report concluded that the error on M_Z would be ± 50 MeV or 550 ppm and that "A factor of 2–3 improvement can be reached with a determined effort" [3]. It was thus generally believed that the error on M_Z represented the limiting factor in the precision with which theoretical predictions could be made. The situation has changed adiabatically, however, and the relative error on M_Z now approaches that of G_F.

Since LEP, as a machine, is not a radically new design, the lesson that we should take is that it is extremely difficult to predict the accuracy with which physical quantities will be measured, even in the relatively short term, and that one should constantly strive to reduce such errors to the minimum level consistent with the available technology. The possibility of precision physics at a muon collider[4] serves to emphasize this point.

With this in mind, and given the great cost and effort that was expended in reducing the error on M_Z to its current value, it is reasonable to look again at G_F and see what is required to reduce its error to a level where it can never become an obstacle limiting the accuracy with which theoretical predictions can be made.

G_F is extracted from the measured value of the muon lifetime, $\tau_\mu = (2.19703 \pm 0.00004)\,\mu\mathrm{s}$ [1] and on the experimental side this is currently the source of the dominant error. New experiments are planned at the Brookhaven National Laboratory, the Paul Scherrer Institute and the Rutherford-Appleton Laboratory and it is likely that the uncertainty on G_F from this source will be reduced to somewhere in the range 0.5–1 ppm.

Most of the work reported here appears in ref.s [5,6].

2 The Fermi Coupling Constant

As given above the current relative error on the Fermi constant is $\delta G_F/G_F = 1.7 \times 10^{-5}$. Of this 0.9×10^{-5} is experimental and 1.5×10^{-5} is theoretical being an estimate of unknown 2-loop QED corrections.

G_F is related to the measured muon lifetime, τ, by the formula

$$\frac{1}{\tau_\mu} \equiv \Gamma_\mu = \Gamma_0(1 + \Delta q). \tag{1}$$

where

$$\Gamma_0 = \frac{G_F^2 m_\mu^5}{192\pi^3} \tag{2}$$

as calculated using the Fermi theory in which the weak interactions are described by a contact interaction. Δq encapsulates the higher order QED corrections and may written as a perturbation series in $\alpha_r = e_r^2/(4\pi)$, the renormalized electromagnetic coupling constant. Thus

$$\Delta q = \sum_{i=0}^{\infty} \Delta q^{(i)} \tag{3}$$

in which the index i gives the power of, α_r that appears in $\Delta q^{(i)}$. Note that Eq.(1) differs from the usual formula[1] in ways that begin to become important at the part-per-million level. It is known[7,8] that

$$\Delta q^{(0)} = -8x - 12x^2 \ln x + 8x^3 - x^4 \tag{4}$$

$$\Delta q^{(1)} = \left(\frac{\alpha_r}{\pi}\right)\left(\frac{25}{8} - 3\zeta(2)\right) + \mathcal{O}(\alpha_r x \ln x) \tag{5}$$

where $x = m_e^2/m_\mu^2$ and ζ is the Riemann zeta function with $\zeta(2) = \pi^2/6$. That the $\Delta q^{(i)}$ remain finite in the limit $m_e \to 0$ is a consequence of the Kinoshita-Lee-Nauenberg theorem[9] whose discovery was largely prompted by this particular observation.

Although the Fermi theory is not renormalizable, the $\Delta q^{(i)}$ can be shown[10] to be finite for all i. This remarkable feature follows from the fact that the $V - A$ interaction is invariant under a Fierz rearrangement that interchanges the wavefunctions of the electron and the muon neutrino. Thus Fermi theory is equivalent to an effective theory in which the muon and electron occupy the same fermion current in the weak interaction lagrangian. After fermion mass renormalization is performed the divergences in the vector part of this current are independent of the fermion mass and hence cancel in exactly the way they would for QED. The lagrangian of Fermi theory is invariant under the transformations $\psi_e \to \gamma_5 \psi_e$ and $m_e \to -m_e$. The QED corrections to the axial vector of the part can thus be obtained from the those of the vector part by changing the sign of the electron mass and hence are finite as well. Moreover the two sets of corrections are equal in the limit $m_e \to 0$ and, in that case, calculations need only be performed using the vector part of the Fermi interaction. This conclusion holds under any regularization prescription and avoids the complications associated with the use of γ_5 in dimensional regularization[11].

The foregoing discussion does not apply to the β-decay of the neutron where the Fierz rearrangement generates scalar and pseudoscalar terms that bear no resemblance to QED and the radiative corrections are consequently not finite.

3 The 2-loop QED Corrections to the Muon Lifetime

The complete 2-loop QED corrections to the muon lifetime require the calculation of matrix element for the processes, $\mu^- \to \nu_\mu e^- \bar{\nu}_e$, $\mu^- \to \nu_\mu e^- \bar{\nu}_e \gamma$, $\mu^- \to \nu_\mu e^- \bar{\nu}_e \gamma\gamma$ and $\mu^- \to \nu_\mu e^- \bar{\nu}_e e^+ e^-$ with up to two virtual photons. All processes contain infrared (IR) divergences coming from either virtual photons, soft bremsstrahlung or both. The cancellation of IR divergences occurs between the various processes but this complication may be avoided by exploiting cutting relations and calculating the 2-loop corrections as imaginary parts of 4-loop diagrams, some of which are shown in Fig.s 1 and 2. In these Feynman diagrams thick lines represent a muon and the thin lines represent either the electron or the neutrinos all of which are taken to be massless. Since the external muon is on-shell any cut passing through a muon line will vanish and the only cuts contributing to the imaginary part are precisely the ones that generate the diagrams appearing in the calculation of muon decay.

Recursion relations [12] obtained by integration-by-parts were first applied to reduce all dimensionally regularized integrals to a small set of relatively simple integrals. The well-behaved primitive integrals were then calculated by taking the external muon momentum, q, off mass shell to obtain expressions as power series in $x = -q^2/m_\mu^2$ and logarithms of x using well-established large mass expansion techniques [13]. As the large mass expansion proceeds many terms, such as those that are topologically tadpoles, can be immediately discarded since they do not give rise to imaginary parts. Since the final result is required for $x = 1$ the complete series must be summed which can now be done in closed form in terms of polygamma functions and certain classes of multiple nested sums [14].

All diagrams were calculated in a general covariant gauge for the photon field and exact cancellation in the final result of the dependence on the gauge parameter was demonstrated.

3.1 Photonic Corrections

Examples of photonic diagrams which when cut give rise to contributions to the muon lifetime at $\mathcal{O}(\alpha^2)$ are shown in Fig.2.

The result obtained for the complete set of photonic diagrams is

$$\Delta\Gamma^{(2)}_{\gamma\gamma} = \Gamma_0 \left(\frac{\alpha_r)}{\pi}\right)^2 \left(\frac{11047}{2592} - \frac{1030}{27}\zeta(2) - \frac{223}{36}\zeta(3) + \frac{67}{8}\zeta(4) + 53\zeta(2)\ln(2)\right) \tag{6}$$

$$= \Gamma_0 \left(\frac{\alpha_r}{\pi}\right)^2 3.55877 \tag{7}$$

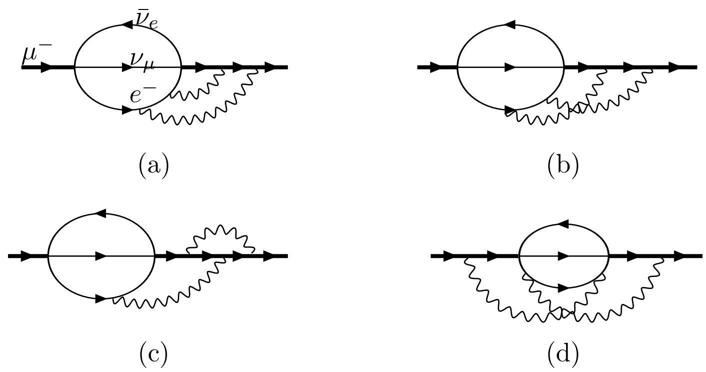

Figure 1: Examples of diagrams whose cuts give contributions to $\mu^- \to \nu_\mu e^- \bar{\nu}_e$, $\mu^- \to \nu_\mu e^- \bar{\nu}_e \gamma$ or $\mu^- \to \nu_\mu e^- \bar{\nu}_e \gamma\gamma$.

where $\zeta(3) = 1.20206...$ and $\zeta(4) = \pi^4/90$.

3.2 Electron-Loops and e^+e^- Pair Creation

Diagrams containing an electron loop whose cuts give contributions to muon decay are shown in Fig.2. The result obtained for these diagrams is

$$\Delta\Gamma^{(2)}_{\text{elec}} = -\Gamma_0 \left(\frac{\alpha_r}{\pi}\right)^2 \left(\frac{1009}{228} - \frac{77}{36}\zeta(2) - \frac{8}{3}\zeta(3)\right) \tag{8}$$

$$= \Gamma_0 \left(\frac{\alpha_r}{\pi}\right)^2 3.22034. \tag{9}$$

The value given in Eq.(9) is consistent with a numerical study carried out by Luke *et al.*[15] in the context of semi-leptonic decays of heavy quarks.

In order to obtain a UV finite answer the a diagrams in which the electron loop is replaced by the photon 2-point counterterm must be included and therefore a decision has to taken as to the renormalization scheme that is to be adopted. This will be discussed further in section 4.

3.3 Hadronic Contributions

Hadronic effects enter τ_μ at the 2-loop level through the diagrams shown in Fig.3. The shaded blob represents the hadronic vacuum polarization of the

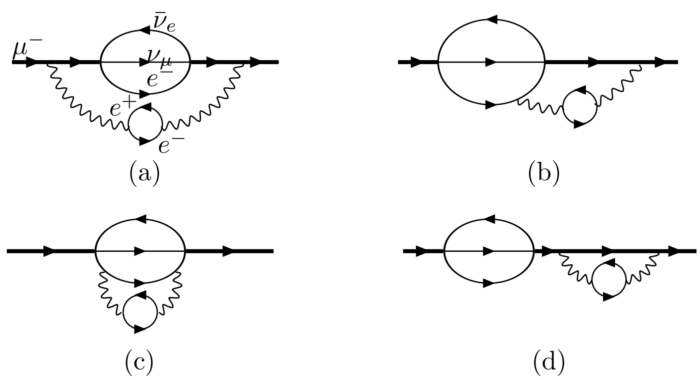

Figure 2: Diagrams containing an electron loop whose cuts give contributions to muon decay, $\mu^- \to \nu_\mu e^- \bar{\nu}_e$, $\mu^- \to \nu_\mu e^- \bar{\nu}_e \gamma$ or $\mu^- \to \nu_\mu e^- \bar{\nu}_e e^+ e^-$.

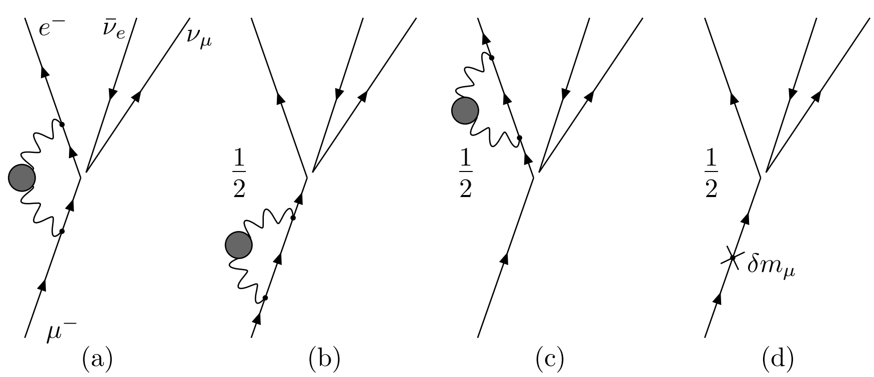

Figure 3: Hadronic contributions to muon decay after Fierz rearrangement of the contact interaction.

photon. The hadronic contribution can be calculated in the usual way using dispersion relations but, in contrast to other well-known situations for which such effects have been calculated [16,17], the momenta of the external fermions, to which the virtual photon is attached, is not fixed. Here the electron participates in the phase-space integration which complicates matters somewhat. Overall the shift induced in the inverse lifetime, Γ_μ, of the muon is given as a convolution integral

$$\Delta\Gamma_{\rm had} = \frac{\alpha_r}{3\pi}\int_{4\rho}^{\infty}\frac{dz}{z}R(m_\mu^2 z)\,\Delta\Gamma(z) \tag{10}$$

over the hadronic spectrum, $R(q^2) \equiv \sigma_{\rm had}/\sigma_{\rm point}$, and in which $\rho = m_\pi^2/m_\mu^2 = 1.61395...$ The convolution kernel, $\Delta\Gamma(z)$ is obtained exactly as an analytic function [5]. When the integral is performed using actual hadronic data the result is

$$\Delta\Gamma_{\rm had} = -\Gamma_0\left(\frac{\alpha_r}{\pi}\right)^2(0.042 \pm 0.002) \tag{11}$$

which includes a rather conservative estimate of the hadronic uncertainty. Still the latter amounts to only 2 parts in 10^8 and so is well under control.

The integral (10) can be used to obtain an expression for the contribution from diagrams where the hadronic vacuum polarization has been replaced by muon loop by setting

$$R(m_\mu^2 z) = \left(1+\frac{2}{z}\right)\sqrt{1-\frac{4}{z}}. \tag{12}$$

which gives

$$\Delta\Gamma_{\rm muon} = \Gamma_0\left(\frac{\alpha_r}{\pi}\right)^2\left(\frac{16987}{576}-\frac{85}{36}\zeta(2)-\frac{64}{3}\zeta(3)\right) \tag{13}$$

$$= -\Gamma_0\left(\frac{\alpha_r}{\pi}\right)^2 0.0364333. \tag{14}$$

The result agrees with that obtained by perturbative methods. The effect of tau loops can be obtained in a similar way and, as expected on the basis of the decoupling theorem, is negligibly small.

4 The Renormalized Electromagnetic Coupling Constant, α_r

The use of dispersion relations to calculate the hadronic and muon loop contributions in the previous section naturally invokes a subtraction of the photon vacuum polarization at $q^2 = 0$ and is therefore equivalent to the on-shell renormalization scheme. In cases where there are two or more widely separated scales, such as m_e and m_μ, use of the $\overline{\mathrm{MS}}$ renormalization scheme is indicated since it automatically incorporates the large logarithms that arise into the value of the renormalized coupling constant, α_r, at tree level.

It is therefore appropriate here to adopt the $\overline{\mathrm{MS}}$ renormalization scheme. The hadronic contributions of section 3.3 that were obtained via dispersion relations must be corrected to convert them from the on-shell to $\overline{\mathrm{MS}}$ renormalization scheme. As it turns out the contribution from muon loops is the same in both schemes when the 't Hooft mass is taken set to $\mu = m_\mu$ as is appropriate here. It can be shown[18] that the $\overline{\mathrm{MS}}$ renormalization scheme is implemented in a a consistent manner by using the results of section 3.3 as they are given and setting

$$\alpha_r = \alpha_e(m_\mu) \equiv \frac{\alpha}{1 - \frac{\alpha}{3\pi} \ln \frac{m_\mu^2}{m_e^2}} + \frac{\alpha^3}{4\pi^2} \ln \frac{m_\mu^2}{m_e^2}. \tag{15}$$

where the logarithm of $\mathcal{O}(\alpha^3)$ was first calculated by Jost and Luttinger[19]. The substitution (15) correctly resums logarithms of the form $\alpha^n \ln^{n-1}(m_\mu^2/m_e^2)$ for all $n > 0$ and incorporates those of $\alpha^3 \ln(m_\mu^2/m_e^2)$. Upon evaluation

$$\alpha_e(m_\mu) = 1/135.90 = 0.0073582.$$

5 Conclusions

It has been over 40 years since the 1-loop QED corrections to the muon lifetime were calculated. The 2-loop contributions have had to await the development of new theoretical techniques, as well substantial increases in computer speed and storage capacity, but are now available.

The complete 2-loop QED contribution to the muon lifetime in the Fermi model may be encapsulated in the quantity $\Delta q^{(2)}$, as defined in Eq.s(1) and (3), and is given by

$$\Delta q^{(2)} = \left(\frac{\alpha_e(m_\mu)}{\pi}\right)^2 \left(\frac{156815}{5184} - \frac{1036}{27}\zeta(2) - \frac{895}{36}\zeta(3) + \frac{67}{8}\zeta(4) + 53\zeta(2)\ln 2 - (0.042 \pm 0.002)\right). \tag{16}$$

This translates into a new value for the Fermi coupling constant of

$$G_F = (1.16637 \pm 0.00001) \times 10^{-5}\,\mathrm{GeV}^{-2} \qquad (9\,\mathrm{ppm})$$

The error has been halved relative to its previous value and is now entirely experimental.

New measurements of the muon lifetime are planned at the Brookhaven National Laboratory, the Paul Scherrer Institute and the Rutherford-Appleton Laboratory and it is therefore likely that the uncertainty on G_F from this source will be reduced to somewhere in the range 0.5–1 ppm.

In that case the theoretical error should still be negligible but other issues such, as error on the muon mass, m_μ, and the upper limit on the muon neutrino mass, m_{ν_μ}, need to be considered.

Finally many of the results and techniques employed here can be readily taken over and applied to inclusive decays of the b-quark.

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