Status of the BFKL Resummation Program

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I discuss the calculation of the next-to-leading logarithmic (NLL) corrections to the BFKL resummation, as well as some of the issues that arise in this formalism at NLL. In particular I consider the large size and apparent instability of the corrections, and I address some of the attempts to understand and tame them.

I. INTRODUCTION TO NLL BFKL

Early last year, after many years of hard work involving many participants, Fadin and Lipatov [1] presented the NLL corrections to the BFKL equation. Since then, there has been much lively discussion of the interpretation of these corrections. In this talk I will discuss some of the results of this activity. I will not address phenomenology here, but some applications of the BFKL resummation are inclusive dijet production at large rapidity separation $y = \ln \hat{s}/(p_{1\perp}p_{2\perp})$, forward jet production in deep-inelastic scattering (DIS), DIS structure functions at small-x, and others.

In order to discuss the NLL corrections it is useful to consider first the solution to the BFKL equation at LL [2]. The BFKL equation is used to resum all powers of $\alpha_s \log(\hat{s})$ in the cross section. This leads to the familiar prediction that at very high energies the cross section scales as a power of the energy:

$$\hat{\sigma} \ \approx e^{Ay} \approx \ \hat{s}^A \ . \tag{1}$$

The quantity $(1 + A)$ is often referred to as the BFKL Pomeron intercept. This scaling behavior is obtained from the solution to the BFKL equation, which is given at LL by the integral

$$f(y, p_{1\perp}, p_{2\perp}) \ = \ \frac{1}{2\pi i} \int_{1/2-i\infty}^{1/2+i\infty} d\gamma \, (p_{1\perp}^2)^{\gamma-1} \, (p_{2\perp}^2)^{-\gamma} e^{\bar{\alpha}_s \chi^{(0)}(\gamma) y} \ , \tag{2}$$

where $\bar{\alpha}_s = \alpha_s N_c/\pi$, and we have performed an azimuthal average over the transverse momenta for convenience. The function

$$\chi^{(0)}(\gamma) \ = \ 2\psi(1) - \psi(\gamma) - \psi(1-\gamma) \tag{3}$$

is the eigenvalue of the LL BFKL kernel, where ψ is the logarithmic derivative of the gamma function. Performing the integral in the saddle-point approximation leads to the exponential rise in the cross section (1) with $A = \bar{\alpha}_s \chi^{(0)}(\frac{1}{2}) = 4\bar{\alpha}_s \ln 2$.

At the heart of the BFKL equation, which is used to derive (2), is the kernel $K(p_{1\perp}, p_{2\perp})$, which is an integral operator in transverse momentum space that is used to build up the BFKL ladder. The contributions to the kernel are shown, schematically, at LL and NLL in Fig. 1. At LL each application of the kernel adds one more factor of $\mathcal{O}(\alpha_s \Delta y)$ to the resummation. It is composed of two types of contributions. The first type corresponds to an emission of a real gluon, in the approximation that it is widely separated in rapidity from any other emissions (known as multi-Regge kinematics). The factor of Δy just comes from the integration over the rapidity of this gluon. The second type corresponds to the virtual contributions which are enhanced by the logarithmic factor Δy. The simplest contribution of this second type can be found by considering the one-loop corrections to $gg \to gg$ scattering.

At NLL one also includes terms of $\mathcal{O}(\alpha_s^2 \Delta y)$ with each application of the kernel. They consist of three types, corresponding to: the emission of two gluons nearby in rapidity, the virtual correction to the emission of one gluon in the multi-Regge kinematics, and the subleading purely-virtual corrections. This last contribution can be found by considering the two-loop corrections to $gg \to gg$ scattering. It is the calculation of these three contributions which took many years and many papers to sort out the technical details[1] Although the full kernel has not been checked in a completely independent manner, many of the pieces of the calculation have received independent confirmation. Two particularly significant checks are the calculation of the virtual correction to the gluon emission in multi-Regge kinematics [4], and the compilation of the three NLL terms into a single kernel with the cancellation of all collinear and soft singularities [5].

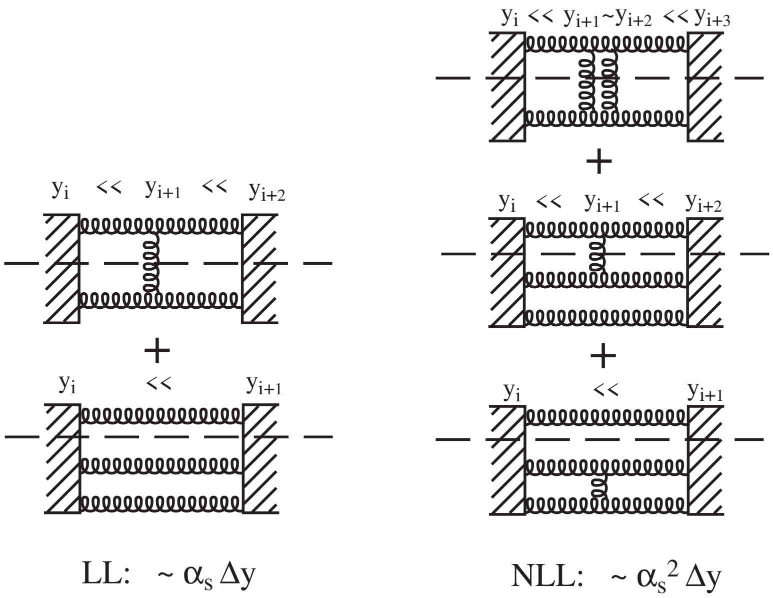

FIG. 1. Contributions to the BFKL kernel at LL and NLL. Gluons that cross the dashed line correspond to real emission.

The final result of this calculation is usually presented by applying the kernel to the LL eigenfunctions, with azimuthal averaging, yielding

$$\begin{aligned}\int d^2p_{2\perp}K(p_{1\perp},p_{2\perp})(p_{2\perp}^2/p_{1\perp}^2)^{\gamma-1} &= \bar{\alpha}_s(\mu)\,\chi(\gamma) \\ &= \bar{\alpha}_s(\mu)\,\chi^{(0)}(\gamma)\left[1-\bar{\alpha}_s(\mu)b_0\ln(p_{1\perp}^2/\mu^2)\right] \\ &\quad+\bar{\alpha}_s^2(\mu)\chi^{(1)}(\gamma)\ , \end{aligned} \tag{4}$$

where $b_0 = 11/12 - n_f/(6N_c)$ and μ is the $\overline{\mathrm{MS}}$ renormalization scale. The NLL correction has been separated into two terms. The first term depends on the scale $p_{1\perp}$ and is associated with the running of the coupling in the LL kernel: $\alpha_s(\mu) \to \alpha_s(p_{1\perp})$. The second term, $\bar{\alpha}_s^2\chi^{(1)}(\gamma)$, is independent of scale and contains the remainder of the NLL corrections [1].

II. PROBLEMS AT NLL.

After completion of the NLL corrections to the BFKL kernel, several issues with the NLL solution quickly became apparent. Depending on one's point of view, these may even signify critical problems for the entire BFKL resummation program. Roughly speaking, they can be separated into issues associated with the running coupling term and issues associated with the scale-invariant term. Although my main focus will be on the scale-invariant term, I briefly touch on the topic of the running coupling term in NLL BFKL.

[1]A list of references can be found in ref. [3], but with no guarantee of completeness.

A. Running coupling problems.

The issues that arise with the running of the coupling in BFKL were not entirely unanticipated. They are related to the fact that the emission of each real gluon causes the momentum carried down the BFKL ladder to diffuse as one moves away from the starting rapidity. It can diffuse to larger values or to smaller values; however, if it diffuses below values around Λ_{QCD} then nonperturbative effects become important, and one can no longer make unambiguous predictions from the perturbative BFKL resummation. This issue was known even before the NLL corrections were completed, although it can be ignored in LL BFKL. This is because logarithms involving two transverse scales do not arise until NLL, and so LL BFKL is calculated at fixed coupling. At NLL, however, this issue can no longer be swept under the rug.

The effects of the running coupling term in the NLL BFKL solution have been considered in several papers. Armesto, Bartels, and Braun [6] considered the modification of the eigenvalues and the eigenfunctions due to the NLL terms in the kernel. They found that, due to the running coupling term in the kernel, the NLL eigenvalues can take on any value along the real axis. This is unlike at LL, where the eigenvalues (3) have a maximum and the equation (2) is well-defined. Thus, the interpretation of the NLL corrections is not entirely well-defined in this approach. Correspondingly, the NLL eigenfunctions contain pieces displaying non-perturbative behavior.

The effects of the running coupling term in the NLL BFKL solution were also included through a different approach by Kovchegov and Mueller [7]. They obtained a NLL solution by explicitly iterating the NLL corrections to the kernel, starting from the LL solution evaluated in the saddle point approximation. They found that the running coupling term leads to a non-Regge behavior in the energy dependence of the cross section. (This was also shown in ref. [8]). This non-Regge behavior is exhibited as a term of the form $\alpha_s^5 y^3$ in the exponential of eq. (1) at high energies. In addition, they showed that the nonperturbative effects from resumming the logarithms in the running coupling should be small as long as $14\zeta(3) b_0^2 \bar{\alpha}_s(\mu)^3 y \lesssim 1$.

From these results, it appears that for some region of kinematics the nonperturbative effects should be small; however, the proper way to deal with them at NLL is not yet completely clear.

B. Scale-invariant problems.

Whereas the problems at NLL due to the running coupling were anticipated to some degree, the problems due to the scale-invariant term were a big surprise. The first indication of this problem was seen immediately by Fadin and Lipatov. The corrections to the leading eigenvalue are large and negative! If we ignore the running coupling term, we obtain

$$\bar{\alpha}_s \chi(\tfrac{1}{2}) \;=\; 2.77\bar{\alpha}_s - 18.34\bar{\alpha}_s^2 \,, \tag{5}$$

for three active flavors. At the not-unreasonable value of $\alpha_s = 0.16$ the NLL corrections exactly cancel the LL term, while for larger values of α_s the eigenvalue becomes negative. Naively, this would indicate that the BFKL Pomeron intercept also becomes negative, leading to a cross section that decreases, rather than increases, as a power of the energy.

Of course, this interpretation relies on the saddle-point evaluation of the NLL generalization of the BFKL solution (2). Upon closer analysis Ross [9] showed that the NLL eigenvalue function $\chi(\gamma)$ no longer has a maximum at $\gamma = \frac{1}{2}$, but has a minimum with two maxima occuring symmetrically on either side of this point[2]. Performing a higher-order expansion of $\chi(\gamma)$, Ross found a smaller correction to the BFKL Pomeron intercept. However, the solution he obtained

[2] The standard procedure in these analyses is to modify the LL eigenfunctions used in eq. (4) in order to make the eigenvalues manifestly symmetric under $\gamma \to 1-\gamma$, following ref. [1].

was not positive definite. It contained oscillations as one varied $p_{1\perp}$ and $p_{2\perp}$. This led Levin [8] to declare that NLL BFKL has a serious pathology.

One might wonder whether the approximate evaluation of the integral performed by Ross is adequate at this stage. Perhaps an exact evaluation is necessary. However, negative cross sections have also arisen when the resummed small-x anomalous dimensions, obtained from the NLL BFKL solution, were used to study DIS scattering at small-x [10]. In any event the NLL corrections to the BFKL solution are large, leading one to question the stability and applicability of the BFKL resummation procedure in general.

III. ATTEMPTS TO FIX/UNDERSTAND THE LARGE NLL CORRECTIONS.

In this section I will discuss several attempts to understand the origin of the large NLL corrections and to control them. The first two proposals, although very different in implementation, both can be traced to correlations that arise when two neighboring gluons are emitted close to each other in rapidity [11]. Essentially, the LL BFKL equation greatly overestimates the contribution of this collinear configuration because of the lack of ordering in transverse momentum.

The first proposal by Salam [12] was to resum the double transverse logarithms of the form $\bar{\alpha}_s \ln^2(p_{1\perp}^2/p_{2\perp}^2)$. This idea is based on the studies of Camici and Ciafaloni [5] on the energy-scale dependence of NLL BFKL. Instead of choosing the symmetric rapidity $y = y_2 - y_1 = \ln \hat{s}/(p_{1\perp}p_{2\perp})$ as the large logarithm to resum, one could equally well have chosen $y^+ = \ln x_2^+/x_1^+ = \ln \hat{s}/p_1^2$ or $y^- = \ln x_1^-/x_2^- = \ln \hat{s}/p_2^2$, where $x_i^\pm$ is the momentum fraction along the positive or negative light-cone for the emitted gluon i. Although these choices are all equivalent at LL, at NLL a change in the logarithm produces a change in the NLL kernel and can introduce the double transverse logarithms.

Motivated by DGLAP-type resummation [13] one finds that the appropriate choice is to resum y^+ when $p_{1\perp}^2 \gg p_{2\perp}^2$ and y^- when $p_{2\perp}^2 \gg p_{1\perp}^2$. In refs. [5] and [1] it was shown that changing from y^+ to y shifts the NLL eigenvalue by terms with $1/\gamma^3$ singularities. Similarly, changing from y^- to y shifts the NLL eigenvalue by terms with $1/(1-\gamma)^3$ singularities. Both the $1/\gamma^3$ and the $1/(1-\gamma)^3$ singularities can be identified in $\chi^{(1)}(\gamma)$, and methods for resumming these singularities were given in ref. [12].

Results of this resummation of double transverse logarithms are shown in Fig. 2, where the leading eigenvalue $\bar{\alpha}_s\chi(\frac{1}{2})$ and its second derivative are plotted as a function of $\bar{\alpha}_s$. The different schemes 1–4 give some measure of the ambiguity in this resummation procedure. In general the eigenvalue is found to be positive after resummation, although less than at LL. In addition the point $\gamma = \frac{1}{2}$ remains a maximum over a wider range of α_s, especially in schemes 3 and 4.

The physical implications of this proposed solution can be seen by further investigating the relation between resummation in $y^\pm$ and y. When $p_{1\perp}^2 \gg p_{2\perp}^2$, the resummation in y^+ requires the ordering $x_2^+ > x_1^+$. Translating back into the symmetric variable y, this implies $y_2 - y_1 > \ln(p_{1\perp}/p_{2\perp})$. Similarly, when $p_{2\perp}^2 \gg p_{1\perp}^2$, the resummation in y^- requires the ordering $x_1^- > x_2^-$, implying $y_2 - y_1 > \ln(p_{2\perp}/p_{1\perp})$. These constraints hold for any two successively emitted gluons. Therefore, the resummation of the double transverse logarithms corresponds to imposing a $p_\perp$-dependent cut, $y_{i+1} - y_i > |\ln(p_{i\perp}/p_{i+1\perp})|$, on the separation in rapidity between the neighboring gluons.

This leads to the second proposal [3] (first suggested in [11] and [14]) for dealing with the large corrections to BFKL at NLL. It is to introduce explicitly a rapidity separation parameter Δ into the BFKL equation, enforcing the condition $y_{i+1} - y_i > \Delta$, where Δ is assumed to be much less than the total rapidity interval y. This parameter can be included systematically at any order in the resummation, and it plays a role for the rapidity resummation similar to the role played by the $\overline{\text{MS}}$ renormalization scale μ for the resummation of logarithms in the running coupling. A change in Δ shifts pieces of the calculation between LL and NLL, such that any differences are always next-to-next-to-leading logarithm (NNLL). Thus, the dependence on Δ can be regarded as an estimate of the uncertainty due to NNLL corrections.

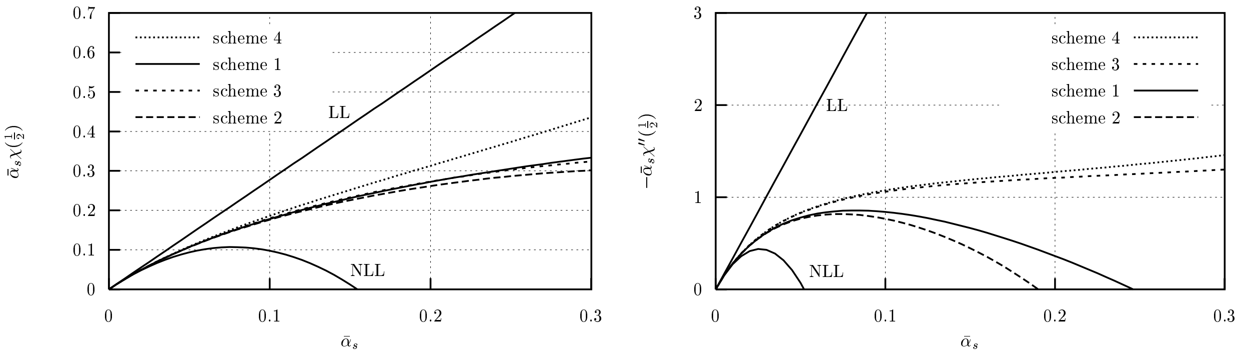

FIG. 2. The result of resumming double transverse logarithms on χ and its second derivative, from Ref. [12].

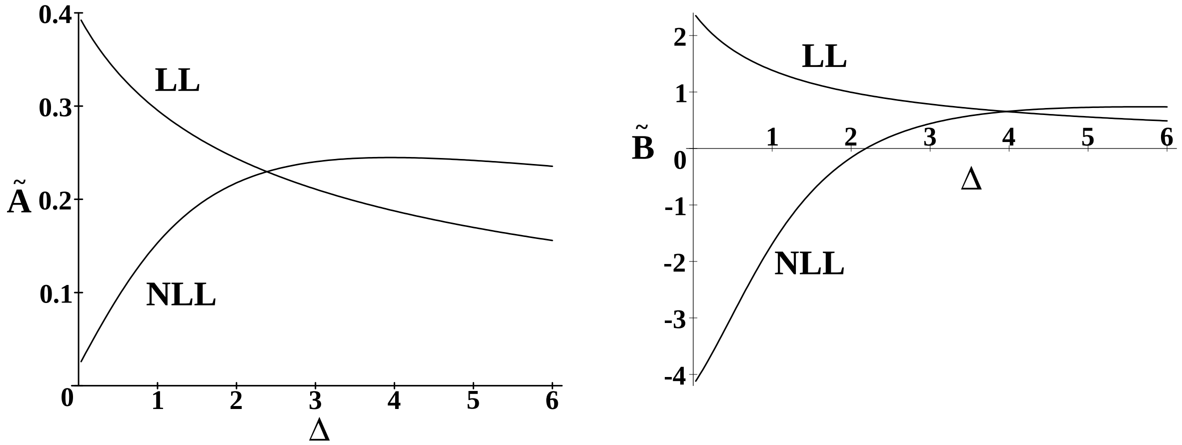

FIG. 3. Dependence of $\tilde{A} = \bar{\alpha}_s\chi(\frac{1}{2})$ and $\tilde{B} = -\frac{1}{2}\bar{\alpha}_s\chi''(\frac{1}{2})$ on Δ for $\alpha_s = 0.15$, from Ref. [3].

Figure 3 shows the dependence on Δ of the leading eigenvalue and its second derivative at LL and NLL for $\alpha_s = 0.15$. We note that the corrections to $\bar{\alpha}_s\chi(\frac{1}{2})$ are not large for $\Delta \gtrsim 2$ and have weak dependence on Δ for large Δ. Also, the point $\gamma = \frac{1}{2}$ is a maximum for this coupling as long as $\Delta \gtrsim 2.2$. Thus, the BFKL resummation is stable for large enough Δ. We also note that this procedure gives predictions of $\bar{\alpha}_s\chi(\frac{1}{2})$ and $\bar{\alpha}_s\chi''(\frac{1}{2})$ for large Δ which are similar to the previous proposal. However, the implications of a large value of Δ for the phenomenological use of BFKL is open to interpretation.

Recently, Forshaw, Ross, and Sabio Vera [15] have studied the inclusion of both the double transverse logarithm resummation and the rapidity veto simultaneously. Whereas ref. [3] emphasized the weak dependence on Δ at large Δ, they were more concerned by the strong dependence at small Δ. They showed that after including the resummation of the double transverse logarithms, the dependence on the rapidity veto parameter Δ was significantly reduced. Given the discussion above, this is reasonable since both the double transverse logarithm resummation and the rapidity veto incorporate the same physical effect: a suppression of gluon emissions close by in rapidity.

A third proposal to deal with the large NLL corrections was presented by Brodsky *et al.* [16]. They re-evaluated the NLL corrections in a suitable physical renormalization scheme, and then used the BLM procedure [17] to find the optimal scale setting for the QCD coupling. The physical connection of this proposal to the other two is less obvious; however, as in the previous proposals it works by reducing the LL prediction (in this case by the choice of the large scale dictated by BLM) combined with a subsequent reduction in the NLL corrections. In addition it yields a very weak dependence on the gluon virtuality $p_{1\perp}^2$ and leads to an approximate conformal invariance.

IV. CONCLUSIONS.

In this talk I have given a brief overview of the BFKL resummation program, and have discussed some of the issues that have arisen from the incorporation of the NLL corrections. The surprisingly large size of the corrections at NLL, as well as the subtle issues related to the running of the coupling, have spurred investigations which will lead to a better understanding of the physics of QCD at high energies. Clearly, this is a challenging and lively field of theoretical research which will significantly impact our understanding of QCD.

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