Center Vortices

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We review recent progress in understanding the center-vortex picture of confinement for $SU(N)$. A notable development is the spectacular confirmation on the lattice of center-vortex predictions for topological confinement and the area law in the fundamental-representation Wilson loop. Recent theoretical progress includes: large-N scaling, the heavy-quark potential for $SU(3)$ baryons, the breakable "string" potential in the adjoint representation, the existence and role of nexuses (solitons which are meeting-points for center vortices whose total flux is zero (mod N), and the role of nexuses in understanding confinement in the d=3 Georgi-Glashow model.

I. INTRODUCTION

The center-vortex picture of confinement is now about twenty years old [1–5]. It has recently come to prominence because of lattice calculations [6,7] which, by various means, compute only the long-range (confining) center-vortex contribution to the fundamental-representation Wilson loop. It turns out that this center-vortex-only area law agrees exactly (that is, to the accuracy of the computation) with the conventionally-computed area law. In other words, the *only* source of confinement is center vortices. Furthermore, it has been shown both theoretically [8,9] and computationally [10] that on the lattice removal of center vortices destroys confinement. The demonstrations proceed, as in effect also do the lattice computations showing center-vortex confinement, by dividing an $SU(N)$ gauge theory into $SU(N)/Z_N$ and Z_N degrees of freedom; the latter are identified with center vortices (see below) and are shown to confine. The mechanism of confinement is essentially a topological one [1], in which closed vortices (of co-dimension two) when linked with the Wilson loop in three or four dimensions yield a phase-factor contribution to the Wilson loop which is an element of the center raised to a power which is the Gauss linking number of the vortex and the loop. The rest of the degrees of freedom contribute only to the perimeter law. No statement need be made, as in some other mechanisms of confinement, about a choice of gauge (although a choice of gauge may be convenient for, *e.g.*, certain lattice computations). Figure 1 shows some results [9] for the $SU(3)$ heavy-quark potential saving only the phase factor of center vortices, compared to the same potential computed in the full theory. The potentials differ at short distances, as is expected (see the discussion on the adjoint potential in Section III).

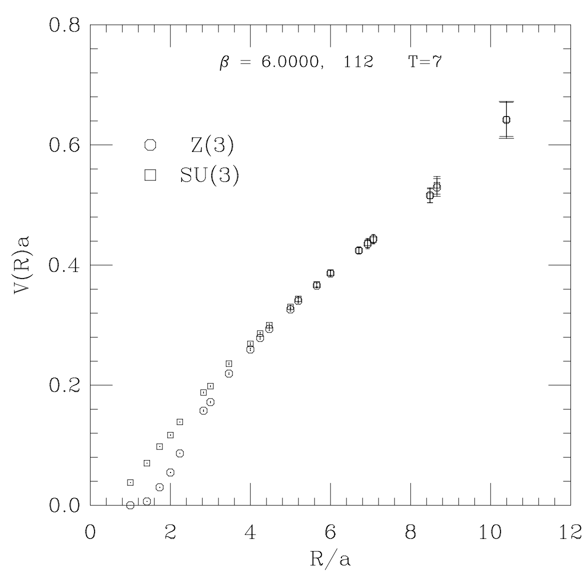

FIG. 1. Heavy-quark potential for smoothed center vortices in $SU(3)$ (circles marked Z(3)), compared to the full theory[from Kovács and Tomboulis [9]].

Over the years several other aspects of the center-vortex picture have been investigated. These include the structure of the center-vortex contribution to the adjoint potential [11,12], which is (as first computed on the lattice by Bernard [13]) approximately linearly-rising for a while, then breaking when the energy stored in the potential is enough to materialize the two gluons needed to screen the two sides of the adjoint Wilson loop. As essential ingredient in this behavior of the adjoint potential is the fact that center vortices have a finite thickness, as determined by a dynamical gluon mass.

It has also been argued [14] that, in $SU(N)$, the potential for N heavy quarks forming a baryon is the sum of N pairwise potentials, each having $1/(N-1)$ the strength of the $q\bar{q}$ potential, rather than an N-body potential with "strings" meeting at a central point. (The author knows of one lattice computation [15] of the baryonic potential for $SU(3)$, which agrees with the center-vortex result; it would be valuable to repeat this computation.)

The continuum center-vortex picture developed in Ref. [1] has been shown [16] to admit a new kind of soliton, the nexus. The nexus, as its name implies, is a meeting point of up to N vortices in $SU(N)$, provided that the total magnetic flux adds to zero (mod N). It is the closest thing to a true monopole in a gauge theory with no symmetry breaking.

There are, of course, competing arguments for confinement (*e.g.*, dual superconductivity, maximal Abelian projection/monopoles, etc.) which we will not review here. But in one particular case, the d=3 Georgi-Glashow model, it has proven possible to clarify the relation of a previous argument by Polyakov [17] for confinement to the center vortex picture. Polyakov's argument invoked a condensate of 't Hooft-Polyakov (TP) monopoles as in dual superconductivity; this picture has been widely interpreted as possessing a $U(1)$ disorder appropriate to compact d=3 QED. In fact, as several authors [18,19] have argued, the Georgi-Glashow model for any finite value of its parameters possesses features that are only explicable in terms of the underlying $SU(2)$ disorder. Ambjørn and Greensite [18] speculate, and the present author [19] shows, that there is a smooth transition from a configuration which the author describes as nexuses [16] and center vortices to the TP monopole as the adjoint-Higgs symmetry breaking of the Georgi-Glashow model is turned off (by reducing the Higgs VEV toward zero), such that the underlying physics is essentially that of center-vortex confinement, with TP monopole confinement as a special case.

In the sections below, we describe this progress in more detail.

II. CENTER VORTICES IN THE CONTINUUM

It was pointed out long ago [20] that, because of the infrared instability of non-Abelian gauge theories in three and four dimensions, a dynamical gluon mass (*not* associated with symmetry breaking) must be generated to cure this instability. The infrared-effective action describing such mass generation is just the sum of the usual Yang-Mills action plus a gauged non-linear sigma model action. This action can only be taken seriously in the infrared, because the dynamical mass vanishes rapidly in the ultraviolet, where the action reduces to the usual Yang-Mills term.

This effective action has numerous solitons, among them center vortices, which are rather like Nielsen-Olesen vortices with a special Higgs structure. They have co-dimension two, meaning that in three dimensions they are tube-like, and in four dimensions eggshell-like. For brevity we restrict ourselves to the d=3 case; see the original literature for the d=4 analogs.

A Euclidean three-dimensional center vortex is described, for $SU(2)$, by the (antihermitean) potential

$$A_i(x) = 2\pi(\frac{\tau_3}{2ig})\epsilon_{ijk}\partial_j \oint_V dz_k[\Delta_m(x-z) - \Delta_0(x-z)] \tag{1}$$

where g is the gauge coupling and $\Delta_{m,0}$ is the free Feynman propagator for mass $m, 0$. (The simple generalization to the $N-1$ distinct vortices of $SU(N)$ is given in Ref. [12].) The integral goes over some closed curve V; in the simple Abelian version of equation (1), integrals over open curves lead to short-range singularities, long-range monopole fields, or both. (In Section VII this restriction is removed by considering intrinically non-Abelian vortices.) Note that while each of the two terms on the left-hand side of (1) have Dirac-string singularities, these cancel so that the vortex potential is finite along the curve V. The massless-propagator part of this potential is in fact a (singular) pure-gauge term, and is responsible for confinement.

We can express the quantization constraint that center vortices must have magnetic fluxes lying in the center of the gauge group as follows. Construct the holonomy of the pure gauge, or massless part, of the vortex around a loop , . Link the loop , and the vortex curve V once, in the Gauss sense. Then the holonomy must lie in the center. We give the general formula for a singly-linked vortex of unit flux in $SU(N)$, the massless part of whose gauge potential we call $A_i^{(0)}$:

$$\exp g \oint_\Gamma dx^i A_i^{(0)}(x) = \exp(i2\pi/N). \tag{2}$$

The reason that this holonomy must lie in the center is that a gauge potential when transported around (linked with) a center vortex must be single-valued; of course, the gauge group for the gauge potentials themselves is $SU(N)/Z_N$ so that the center is trivially represented. Equation (2) represents the homotopy $\Pi_1(SU(N)/Z_N) = Z_N$.

Now consider the contribution of center vortices to confinement. Take the fundamental-representation Wilson loop to be *large* in the sense that its length and all other spatial scales are large compared to m^{-1}. Then to compute the area law we can omit the massive propagator in (1). A simple computation [1] gives the result for one vortex V:

$$Tr_F \exp \oint_\Gamma dx_i A_i(x) = \exp\left(\frac{-2\pi i}{2} \oint_V dz_i \oint_\Gamma dx_j \epsilon_{ijk} \frac{(x-z)_k}{4\pi|x-z|}\right). \tag{3}$$

The integral in (3) is the Gauss linking integral, so that the contribution of a center vortex of unit flux linked J times to the Wilson loop is $\exp(2\pi i J/N)$, that is, $(-)^J$ for $SU(2)$. It is straightforward to show [1,12] that an area law follows from averaging the fluctuations in this phase factor. All contributions to the Wilson loop from the massive terms in (1) are perimeter-law.

III. THE ADJOINT POTENTIAL

For the adjoint potential of $SU(2)$ the calculation corresponding to equation (3) yields $\exp(2\pi i J) \equiv 1$, so there is no area law, and no contribution from the long-range pure-gauge part of the center vortex. However, the massive part does contribute. For an adjoint Wilson loop whose size is comparable to m^{-1} the massive, or thick, part of the vortex may only partially overlap the Wilson loop. The result [11,12] is that the adjoint potential decreases as the size of the adjoint Wilson loop decreases. For distances on the scale of m^{-1} the potential rises *roughly* linearly, then reaches an asymptotic value which scales with m. This simply represents the influence of all the vortices within a distance of order m^{-1} of the Wilson loop; vortices remote from the loop give exponentially-vanishing contributions. The interpretation [11–13] of this asymptotic value is that it is the energy required to pop a pair of gluons out of the vacuum to screen the Wilson loop.

In calculating the adjoint Wilson loop one must be careful not to make common approximations such as saving only quadratic terms in the gauge potential A_i in the expansion of the exponent of the Wilson loop. This is analogous to the approximation $\cos x \simeq 1 - x^2/2$; it fails to recognize the fact that the Wilson loop is a periodic function of flux integrals like $\oint dx^i A^i$. Such approximations lead to Casimir scaling (the potential scales with the quadratic Casimir of the representation used in the Wilson loop), but there is no requirement for Casimir scaling in the exact theory.

This has led to some controversy over what happens in the center-vortex picture at large N. Based on experience with d=2 QCD (which does not have center vortices) one might expect that all representations, including the adjoint, show an area law at large N. The present author claims [12] that there is a kind of large-N scaling for center vortices, in which the fundamental-loop string tension behaves like $\rho N/(N-1)$ where ρ is the two-dimensional density of vortices, but in which the adjoint potential always shows only a perimeter law. In fact, it is shown that the leading term in the adjoint potential is a universal function independent of N at large N. But there are different views; see Ref. [21]. These authors invoke center vortices whose size grows logarithmically with N in an attempt to save Casimir scaling and effective adjoint confinement on at least some distance scales. There seems to be no place for such growth of vortex size in the continuum picture discussed in the present work.

We note that old arguments about the suppression of various kinds of solitons at large N, coming from action-barrier factors like $\exp(-A/g^2) \sim \exp(-N)$, do not necessarily hold since this exponential vanishing can be exactly matched by exponential growth coming from collective-coordinate integrals. See Ref. [12] for such arguments in the center-vortex case.

IV. BARYONIC POTENTIALS AND CENTER VORTICES

For years it has been argued, based on a strong-coupling approximation, that in $SU(3)$ the potential between three heavy quarks is equivalent to three strings attaching the quarks to a central line (the so-called Y configuration). But this is not what the center-vortex picture says [1,14]. There is instead a Δ configuration, with a sum of three linearly-rising potentials acting pairwise between the quarks, and with half the strength of the $q\bar{q}$ potential. The center-vortex mechanism is a non-trivial generalization of that described in Section II for the usual Wilson loop. An essential part of the generalization to three (N) quarks in $SU(3)$ ($SU(N)$) is the non-Abelian Stokes' theorem applicable to baryonic Wilson loops, first proved in [14]. One must use this Stokes' theorem to convert loop integrals to surface integrals, in order to demonstrate that a large baryonic Wilson loop in the presence of a center vortex develops a phase factor in the group center, depending on the linkage of the vortices to the baryonic loop. In turn, this linkage must be defined topologically so that it can be reduced to a sum of Gaussian link integrals. (Such a description is, in fact, impossible for an Abelian theory.) Ultimately, the fluctuations in the phase factor lead to the Δ potential described above.

The non-Abelian Stokes' theorem relates a line integral to a surface integral; the surface is bounded in part by the contour of the line integral. The link number of a vortex and a contour is found by counting (with sign) the number of vortex penetrations of this surface. For a baryonic line integral the contour is not a simple closed and oriented loop, and the required surfaces are more complicated than those which span the pairs of quark lines. Such surfaces make no sense, because their bounding contours have no fixed orientation. Instead, the three surfaces involved each have one quark line as part of the contour and a shared central line which closes the contour. Of course, the surface integrals must be independent of this choice of central line, and they are. Because the center vortices have magnetic fluxes lying in the center of the group, it can be shown that this arbitrary central line also has no effect on the value of the linkage of a vortex with the baryonic Wilson loop, and allows for the necessary demonstration of topologically-invariant linkages. What happens is that for $SU(N)$ one only needs to define link numbers mod N, and that is what is described by the non-Abelian Stokes' theorem.

V. NEXUSES

There are generalizations [1,16] of the closed loops of center vortices as described in equation (1). They consist of up to N vortex lines (surfaces) meeting at a point (along a line) in d=3(4), provided that the net magnetic flux is zero mod N. The joining region of these vortices is a soliton of size m^{-1} which has been called [16] a *nexus*. It must be emphasized that these solitons only exist in a non-Abelian gauge theory; naive Abelian versions of nexuses and the related vortex configurations suffer from naked Dirac strings and other diseases.

As sketched in Fig. 1, a nexus in $SU(2)$ is a place where two center vortices as described in equation (1) meet, but with *oppositely*-directed field strengths (this concept is easy to grasp for an Abelian theory, but less so for a non-Abelian theory; nonetheless, it can be made sensible). Given that nexuses exist, vortices may either terminate on them or form closed loops. So if a vortex line begins on a nexus it must end on an anti-nexus, and the only configurations available in $SU(2)$ are closed loops with as many nexuses as antinexuses, and corresponding reversals of the directions of the magnetic fields. These reversals have no influence on the area law, since $\exp \pm i\pi = -1$ and the Wilson-loop arguments given earlier are unchanged.

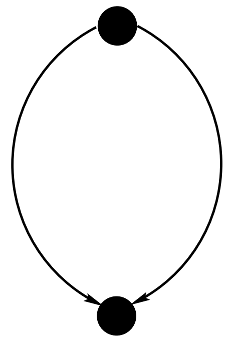

FIG. 2. A sketch of an SU(2) nexus and anti-nexus (black circles) separating regions of center vortices with oppositely-directed field strengths.

Presumably in the pure gauge theory nexuses are suppressed to low levels, because it costs action to make them, which is unlikely to be made up for by the increased entropy of field reversals. But it should be possible to arrange special lattice boundary conditions which would only allow for nexus-vortex combinations to be formed, and it would be interesting to study them as further confirmation of the underlying physics of center vortices.

Furthermore, there are cases where nexuses play an essential role; we describe one such case next.

VI. NEXUSES, VORTICES, AND THE D=3 GEORGI-GLASHOW MODEL

The Georgi-Glashow model is that of an $SU(2)$ gauge theory with adjoint-Higgs symmetry breaking; two "charged" gauge bosons become massive, with mass $M = gv$ where v is the Higgs VEV. The "neutral" or electromagnetic gauge boson is massless (at least classically). This model possesses TP monopoles with long-range magnetic fields. At large distances the TP monopole gauge potential becomes that of the Wu-Yang monopole. Polyakov [17] long ago showed that these monopoles, if condensed, would lead to confinement in d=3 by the dual superconductivity mechanism then in favor. His demonstration is appropriate to the semiclassical regime $v \gg g$, where the action of the TP monopoles was $\mathrm{O}(4\pi M/g^2) \simeq 4\pi v/g \gg 1$. Polyakov showed that a condensate of TP monopoles could form, with an exponentially-small monopole volume density, of order $\exp(-const.v/g)$. This condensate density induced a mass for the (classically massless) neutral field, of the same order.

Consider first an isolated TP monopole, with massless photonic fields. This monopole shows quantized magnetic flux as expressed through the usual integral $\int dS_i B_i$ over the sphere at infinity, where B_i is the 't Hooft magnetic field. This is strictly a non-Abelian effect, since an Abelian magnetic field (*i.e.*, $\vec{B} = \vec{\nabla} \times \vec{A}$) must have, by Stokes' theorem, zero flux through any closed surface. Flux quantization expresses the homotopy $\Pi_2(SU(2)/U(1)) = Z$.

If the photonic field picks up a mass, however small, the flux at infinity must vanish because of the exponential decrease of the fields. In Polyakov's semiclassical limit this decrease is postponed to exponentially-large distances, since $v \gg g$. But at shorter distance scales confinement is by conventional dual superconductivity effects.

What happens as we reduce the Higgs VEV v, in order to turn off the symmetry breaking? It appears that at v=0 we are left with a theory very much like pure gauge theory with no scalar fields (the now-massless scalar fields do not, for example, stabilize the infrared behavior of the theory [19]). In such a case, there will be dynamical mass generation as described in earlier sections of this paper, and the center-vortex picture follows. In fact, because of infrared instability, the threshhold for dynamical mass generation [20] occurs at a critical value v_c of order g [19], where the TP monopole action is no longer large. Now there are *two* sources of mass: The first is the usual Higgs effect, which gives a mass $M = vg \sim g^2$, and the second is dynamical mass generation as described earlier, which gives a mass $m \sim g^2$ equally to all three gauge fields. These two masses come from different mechanisms and are generically unequal.

One can write down an effective action which expresses both of these mass effects; it is simply the usual Georgi-Glashow action supplemented by a gauged non-linear sigma model term, as was used to describe dynamical mass generation. It turns out [19] that this action has nexus-vortex solitons which are in effect TP monopoles with two different masses, one for the charged fields and one for the photonic field. This, as we have seen earlier, is in any event what happens once TP monopoles condense in the semiclassical picture of Polyakov. These solitons reduce to the original TP monopole in the limit $m = 0$, and reduce to a version of the original nexus-vortex solitons found in Ref. [16].

To describe these solitons we start from an Abelian configuration, which is a modification of the center vortex described in equation (1). This configuration is:

$$A_i(x) = 2\pi(\frac{\tau_3}{2i})\epsilon_{ijk}\partial_j[\int_0^{\infty} dz + \int_0^{-\infty} dz]\Delta_m(x-z). \tag{4}$$

It partially describes two Abelian monopoles back-to-back, with oppositely-directed field strengths along the z-axis. By itself it is unacceptable, because there are no long-range pure-gauge parts which cancel the Dirac string singularities in (4). Figure 2 shows schematically the nature of the field lines and Dirac strings coming from the gauge potential (4). The fields have an extent transverse to the z-axis of m^{-1}. Because the gauge potential (4) is Abelian, the Dirac strings must be present in order to yield zero net magnetic flux over the sphere at infinity.

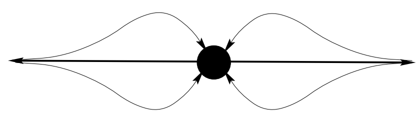

FIG. 3. Sketch of the field lines associated with the gauge potential of equation (4). Thick lines indicate Dirac strings.

In the limit $m = 0$ equation (4) reduces to a Wu-Yang monopole presented in a singular gauge; since the TP monopole also becomes the Wu-Yang monopole at large distances, (4) is suitable to describe the TP monopole at large distances in this limit, provided that the Dirac strings can be removed with a (necessarily non-Abelian and singular) gauge transformation, which serves the role of the Δ_0 term in (1). This gauge transformation does two things simultaneously and non-trivially; the first is to remove the Dirac strings in (4). The second is to provide the correct long-range behavior of the gauge potential so that *both* the Higgs mass term and the dynamical mass term in the action are finite at long distances. The required gauge transformation is [19]:

$$A_i \rightarrow VA_iV^{-1} + V\partial_iV^{-1};\ V = e^{-i\phi\tau\cdot\hat{r}/2} \tag{5}$$

where the gauge potential A_i is given in (4).

The remaining steps are straightforward; details can be found in Ref. [19]. One writes down an *ansatz* for the gauge potential which 1) reduces to the TP monopole in the limit $m = 0$, 2) reduces to the nexus-vortex potential (5) (that is, the gauge transform of (4)) in the limit $M = 0$, and 3) for $m \neq 0$ approaches the gauge $V\partial_iV^{-1}$ at infinity. When neither mass vanishes, this *ansatz* describes a nexus-vortex combination with the nexus essentially describing the short-range behavior of the TP monopole, with long-range fields essentially describing the pure-gauge nexus-vortex combination, and with no Dirac strings. The configuration of fields then looks like those of Figure 3:

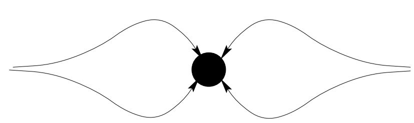

FIG. 4. Sketch of the field lines associated with a non-singular nexus-vortex combination with two gauge masses M, m; it reduces to the TP monopole in the $m = 0$ limit.

The closed-loop version of this configuration, like the pure-gauge nexus-vortex combination of Section V and Figure 1, shows topological confinement in the same manner as described in Section II. Ambjørn and Greensite [18] have also speculated that configurations similar to that shown in Figure 3 are relevant to confinement in the Georgi-Glashow model. These authors discuss other ways in which the d=3 Georgi-Glashow model is not just three-dimensional compact QED. The configuration of Figure 3 is as close as one can get to TP monopoles when the photonic field is massive, and the homotopy is the Π_1 homotopy of vortices and not the Π_2 homotopy of true monopoles.

The upshot is that even for large values of v/g in the Georgi-Glashow model the confining properties of this model at large enough distances are those of the center-vortex picture. One need not turn off the symmetry breaking completely to recover center vortices; at a critical value $v_c \sim g$ of the Higgs VEV this model is qualitatively the same as that of the pure-gauge theory with no Higgs fields.

VII. SUMMARY

In recent times, lattice computations and lattice-based theory have shown that center vortices are the specific and only mechanism of confinement in gauge theory. The center-vortex picture has been extended in several directions, including large-N scaling, the adjoint potential, the baryonic potential in $SU(3)$, the existence of nexuses as interpolators between center-vortex segments of different character, and the role of nexus-vortex combinations in the Georgi-Glashow model.

ACKNOWLEDGEMENTS

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