# String Theory, Matrix Model, and Noncommutative Geometry


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Compactification of Matrix Model on a Noncommutative torus is obtained from strings ending on D-branes with background B field. The BPS spectrum of the system and a novel $SL(2,Z)$ symmetry are discussed.


Noncommutativity of space-time coordinates emerged in string theory recently in the context of coincident D-branes [1]; in fact the embedding coordinates of D-branes turned out to be noncommutative. These noncommutative coordinates in the case of 0-branes are elevated to the dynamical variables of Matrix Theory, which is conjectured to describe the strong coupling limit of string theory, or M-theory, in the infinite momentum frame [2].

Another kind of noncommutativity in spce coordinates has been recently observed in Matrix Theory which is superficially different from the above kind. It comes from the application of the non-commutative geometry (NCG) techniques pioneered by A. Connes to the Matrix Theory compactifications [3].

As a formulation of M-theory, Matrix Theory must describe string theory when compactified on a circle; further compactifications being neccessary to accomodate low energy physics. A class of toroidal compactifications have been known , which relies on a certain commutative subalgebra of matrices [4,5]. The subalgebra being an equivalent description of the manifold of torus on which compactification is performed.

It was observed by Connes, Douglas and Schwarz (CDS) that a nonabelian generalization of this algebraic description of the manifold of compactification, in the spirit of NCG, it is possible to arrive at a different compactification of Matrix-model, with the subsequent novel physical result of appearance of a constant background of the 3-form field in the 11 dimensional supergravity limit.

It was immediately observed by Douglas and Hull [6] that a consequent deformed SYM theory and, therefore indirectly, the noncommutative torus (NCT) compactification is a natural consequence of certain D-brane configurations in string theory. The subject has been pursued in recent works [7,8,9,10,11].

Thus there is a close connection between constant background Kalb-Ramond field B and the nonabelian torus compactification of the Matrix Theory. But, it is not obvious how a background B field can make the coordinates noncommutative and how this noncommutativity differs from that of the coincident D-branes.

We will show explicitly how the CDS noncommutativity arises from D-branes in the presence of B field backgraound and compare it with the noncommutativity due to coincident D-branes [12,13,21]. This noncommutativity persists in higher tori. The dynamical variables of Marix Theory are $N \times N$ matrices which are function of time, with $N$ going to infinity and with the supersymmetric action,

$$I = \frac{1}{2g\sqrt{\alpha'}} \int d\tau \quad Tr \left\{ \dot{X}_a \dot{X}_a + \frac{1}{(2\pi\alpha')^2} \sum_{a<b} [X^a, X^b]^2 + \frac{i}{2\pi\alpha'} \Psi^T \dot{\Psi} - \frac{1}{(2\pi\alpha')^2} \Psi^T \Gamma_a [X^a, \Psi] \right\}. \tag{1}$$

$X^a$, $a = 1, ..., 9$ are bosonic hermitian matrices and $\Psi$ are 16 component spinors. $\Gamma^a$ are $SO(9)$ Dirac matrices. Classical time independent solutions have commuting $X^a$, therefore simultaneously diagonalizable, corresponding to the classical coordinates of $N$ 0-branes. In general off-diagonal elements of $X^a$ correspond to substringy noncommutative structure of M-theory.

Compactification of coordinates $X_i$ of Matrix Theory on a space-like torus of radii $R_i$ has been shown [5] to require existence of the matrices $U_i$ with the property


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$$\begin{array}{c} U_i X_i U_i^{-1} = X_i + R_i \\ U X^a U^{-1} = X^a \qquad a \neq 1,2 \\ U \Psi U^{-1} = \Psi \end{array} \tag{2}$$

Consistency between these equations requires:

$$U_i U_j = e^{i\theta_i j} U_j U_i, \tag{3}$$

for real numbers $\theta_i j$; where for the usual commutative torus, $\theta_i j = 0$. In fact *rational* $\theta_i j$ will also give a commutative torus. It is easily seen that for $\theta = 0$,

$$\begin{array}{c} X_i = i\frac{\partial}{\partial \sigma_i} + A_i \quad ,i = 1, ..., p \\ U_i = e^{i\sigma_i R_i}. \end{array} \tag{4}$$

is a solution of eq. (2) and (3) and its insertion in the action results in the p+1 dimensional SYM on the dual torus. Here $\sigma_i$ parameterize the dual torus.

In the case of two torus, Connes, Douglas and Schwarz [3] observed that in Eq. (3), $\theta$ can be taken different from zero and it corresponds to compactification on a noncommutative torus (NCT) and the resulting gauge theory is the SYM with the commutator of the gauge fields replaced by the Moyal bracket. In NCG the $c^*$ algebra of functions over the manifold, is generalized to a noncommutative $c^*$ algebra [14]. Thus, the algebra generated by the commuting matrices $U_1$ and $U_2$ in the case of usual $T^2$, is generalized to the algebra generated by $U_1$ and $U_2$ satisfying the relation (3), which now defines a "noncommutative" torus, $T^2_\theta$. The solutions of (3) are then,

$$X_i = iR_i\partial_i + A_i, \tag{5}$$

where $A_i$ now are functions of $\tilde{U}_i$, with $\tilde{U}_i$ satisfying

$$\begin{array}{c} \tilde{U}_1\tilde{U}_2 = \quad e^{-i\theta}\tilde{U}_2\tilde{U}_1, \qquad U_i\tilde{U}_j = \tilde{U}_j U_i \\ [\partial_i, \tilde{U}_j] = i\delta_{ij}\tilde{U}_j; \quad i,j = 1,2. \end{array} \tag{6}$$

Substituting them in the action, we get the SYM theory on the NCT dual to the original one, with the essential modification being, the replacement of commutators of gauge fields by the Moyal bracket,

$$\begin{array}{c} \{A, B\} = A * B - B * A, \\ A * B(\sigma) = e^{-i\theta(\partial_1'\partial_2'' - \partial_2'\partial_1'')} A(\sigma')B(\sigma'')|_{\sigma'=\sigma''=\sigma}. \end{array} \tag{7}$$

with $\sigma = (\sigma_1, \sigma_2)$.

The BPS spectrum of the compactified Matrix Theory on the noncommutative torus has been calculated[3,15], and is,

$$\begin{array}{c} E = \frac{R}{n-m\theta}\Big\{ \frac{1}{2}\big(\frac{n_i - m_i\theta}{R_i}\big)^2 + \frac{V^2}{2}\big[m + (n - m\theta)\gamma\big]^2 \\ +2\pi\sqrt{(R_1 w_1)^2 + (R_2 w_2)^2}\Big\}. \end{array} \tag{8}$$

where $V = (2\pi)^2 R_1 R_2$ and $\frac{n_i}{R_i}$ are KK momenta conjugate to $X_i$; $m_i = \epsilon_{ij} m_{j-}$, with $m_{i-}$ winding number of the longitudinal membrane along $X_i$ and $X_-$ direction; $R$ the compactification radius along the $X_-$ direction and $w_i$ are the momenta of BPS states due to the transverse coordinates and are constrained by:

$$w_i = \epsilon_{ij}(nm_j - mn_j). \tag{9}$$

$n$ is the dimension of matrices, $m$ is the winding number of the membrane around torus and $\theta$ is the deformation parameter of the torus. The mass spectrum (8) is invariant under an $SL(2,Z)_N$ generated by

$$\begin{array}{c} \theta \to \frac{-1}{\theta} \\ m \to n \quad , \quad n \to -m \\ m_i \to n_i \quad , \quad n_i \to -m_i \\ \gamma \to -\theta(\theta\gamma + 1) \\ R_i \to \theta^{-2/3} R_i \quad , \qquad R \to \theta^{-1/3} R \end{array} \tag{10}$$

and

$$\begin{array}{ccc} & \theta \to \theta + 1 & \\ n \to n + m & , & m \to m \\ n_i \to n_i + m_i & , & m_i \to m_i . \end{array} \tag{11}$$

This invariance is to be expected on the basis of the NCG considerations. It is the SL(2,Z) invariance of the $c^*$-algebra defining the NCT [3].

We will now see how the above noncommutativit appears in string theory in the presence of D-branes in the $B_{\mu\nu}$ background. The dynamics of strings ending on a p-brane in the background of the antisymmetric field, $B_{\mu\nu}$ is [15],

$$S = \frac{1}{4\pi\alpha'} \int_\Sigma d^2\sigma \big[ \eta_{\mu\nu} \partial_a X^\mu \partial_b X^\nu g^{ab} + \epsilon^{ab} B_{\mu\nu} \partial_a X^\mu \partial_b X^\nu + \frac{1}{2\pi\alpha'} \oint_{\partial\Sigma} d\tau A_i \partial_\tau \zeta^i , \tag{12}$$

where $A_i,\ i = 0, 1, p$ is the $U(1)$ gauge field living on the D-brane and $\zeta^i$ its internal coordinates. The action is invariant under the combined gauge transformation [1]

$$\begin{array}{c} B_{\mu\nu} \to B_{\mu\nu} + \partial_\mu \Lambda_\nu - \partial_\nu \Lambda_\mu \\ A_\mu \to A_\mu - \Lambda_\mu . \end{array} \tag{13}$$

The gauge invariant field strength is then

$$\mathcal{F}_{\mu\nu} = B_{\mu\nu} - F_{\mu\nu} \quad , \quad F_{\mu\nu} = \partial_{[\mu} A_{\nu]} . \tag{14}$$

which leads to the following mixed boundary conditions,

$$\left\{ \begin{array}{c} \partial_\sigma X_0 = 0 \\ \partial_\sigma X_i + \mathcal{F}_{ij} \partial_\tau X_j = 0 \\ \partial_\sigma X_i - \mathcal{F}_{ij} \partial_\tau X_j = 0 \\ \partial_\tau X_a = 0 \quad , \quad a = p, ..., 9. \end{array} \right. \tag{15}$$

Canonical commutation relations of $X_i$ and their conjugate momenta $P_i$, $i = 1, ..., p$:

$$P_i = \partial_\tau X_i - \mathcal{F}_{ij} \partial_\sigma X_j \quad , \tag{16}$$

$$[X^\mu(\sigma, \tau), P^\nu(\sigma', \tau)] = i\eta^{\mu\nu} \delta(\sigma - \sigma') . \tag{17}$$

Lead to the noncommutative center of mass coordinates:

$$x^i = \frac{1}{\pi} \int X^i(\sigma, \tau)\ d\sigma , \tag{18}$$

$$[x_i, x_j] = \pi i \mathcal{F}_{ij} . \tag{19}$$

This noncommutativity of space coordinates is the reason for the noncommutativity which appears in the compactification of Matrix Theory on a torus with a constant 3-form field, which can be seen by going to the string matrix model [16]. To see the connection between the noncommutativity due to the boundary conditions on the d-branes and the noncommutativity which appears in the transverse coordinates of coincident D-branes, recall that D2-branes with a non-zero U(1) gauge field in the background contain a distribution of 0-branes proportional to $\mathcal{F}$ [12,13,17,18,19]; thus the noncommutativity of the coordinates.

It is interesting that the mechanism which produces the original noncommutativity in the description of D-branes, and leads through a set of arguments to the particular form of the commutation relation in (19), is simply derived from the string action (12) in the presence of the $\mathcal{F}$ and mixed boundary conditions with $B$ field background.

We compactify the $X^i$ direction and wrap the 2-brane around the 2-torus and use the center of mass coordinates $x^i$ and their conjugate momenta to construct the generators of the $c^*$ algebra of the noncommutative torus; proving that the compactification, in the presence of $U(1)$ field strength, for D-membrane requires a NCT;

$$
\begin{aligned}
U_1 x^1 U_1^{-1} &= x^1 + R_1 \\
U_2 x^2 U_2^{-1} &= x^2 + R_2 \\
U_i x^j U_i^{-1} &= x^j \qquad i \neq j = 1,2
\end{aligned}
\tag{20}
$$

A solution to these equations is:

$$
\begin{aligned}
U_1 &= exp\{-iR_1\big[a(p_1 - \tfrac{x^2}{\pi\mathcal{F}}) - \tfrac{x^2}{\pi\mathcal{F}}\big]\} \\
U_2 &= exp\{-iR_2\big[a(p_2 + \tfrac{x^1}{\mathcal{F}}) + \tfrac{x^1}{\mathcal{F}}\big]\},
\end{aligned}
\tag{21}
$$

with $a^2 = 1 + \frac{\pi^2\mathcal{F}^2}{R_1R_2}$. The above relations leads to

$$
U_1U_2 = e^{i\pi\mathcal{F}}U_2U_1. \tag{22}
$$

This result reproduces the Matrix Theory compactification on the NCT formulated by CDS, described previously. It was argued there that, the noncommutativity of the torus is related to the non-vanishing of 3-form of M-theory, which in the string theory reduces to the antisymmetric NSNS 2-form field, $B_{\mu\nu}$. In our case noncommutativity of the torus on which the D-membrane of string theory is compactified, is a direct result of the non-vanishing $B$ field. In fact using the Matrix model formulation of string theory [16]. it is straightforward to obtain CDS results.

The noncommutativity of the $c^*$ algebra (20) and (3) of the NCT is similar to, but distinct from, the noncommutativity of the coordinates as in (19) and as it appears in Matrix Theory and bound states of D-branes. The similarities are obvious, but the differences are subtle. In fact it is possible to see that when $\mathcal{F}$ is quantized to a rational number, by an SL(2,Z) transformation, we can make the $U_1$ and $U_2$ commute, i.e. we can make the torus *commutative*, while the coordinates are *noncommutative*. Thus for *irrational* parameter $\theta$, we are dealing with a new form of noncommutativity not encountered in ordinary Matrix theory or in the context of D-brane bound state.

We will now find the BPS spectrum of a system of (D2-D0)-brane bound state. It is convenient to consider the T-dual version of the mixed brane, in which we only need to deal with commutative coordinates and commutative torus, and are able to calculate the related spectrum just by theusual string theory methods.

Applying T-duality in an arbitrary direction, say $X^2$,

$$
\left\{
\begin{array}{c}
\partial_\sigma X^0 = 0 \\
\partial_\sigma(X^1 + \mathcal{F}X^2) = 0 \\
\partial_\tau(X^2 - \mathcal{F}X^1) = 0 \\
\partial_\tau X^a = 0 \quad , \quad a = 3,...,9,
\end{array}
\right.
\tag{23}
$$

describing a tilted D-string which makes an angle $\phi$ with the duality direction, $X^2$:

$$
\cot\phi = \mathcal{F}.
$$

Thus we consider a D-string winding around a cycle of a torus defined by:

$$
\tau = \frac{R_2}{R_1}e^{i\alpha} = \tau_1 + i\tau_2 \quad , \quad \rho = iR_1R_2\sin\alpha + b = i\rho_2 + b, \tag{24}
$$

where $b = BR_1R_2\sin\alpha$ is the flux of the $B$ field on the torus. The D-string is located at an angle $\phi$ with the $R_1$ direction such that it winds $n$ times around $R_1$ and $m$ times around $R_2$. Hence

$$
\cot\phi = \frac{n}{m\tau_2} + \cot\alpha. \tag{25}
$$

The BPS spectrum of this tilted D-string system gets contributions from both the open strings attached to the D-string and the D-string itself. The open strings have mode expansions [16]:

$$
\left\{
\begin{array}{c}
X^i = x_0^i + p^i\tau + L^i\sigma + Oscil. \quad , i = 1,2 \\
X^0 = x_0^0 + p^0\tau + Oscil. \\
X^a = x_0^a + Oscil. \quad , \quad a = 3,...,9
\end{array}
\right.
\tag{26}
$$

where $p^i$ and $L^i$, in usual complex notation, are:

$$p = \ r_1 \frac{n+m\tau}{|n+m\tau|^2}\sqrt{\frac{\tau_2}{\rho_2}} \qquad ; r_1 \in Z. \tag{27}$$

$$L = \ q_1 \frac{\rho(n+m\tau)}{|n+m\tau|^2}\sqrt{\frac{\tau_2}{\rho_2}} \qquad ; q_1 \in Z. \tag{28}$$

Mass of the open string is then,

$$M^2 = |p+L|^2 + \mathcal{N} = \frac{\tau_2}{|n+m\tau|^2}\frac{|r_1+q_1\rho|^2}{\rho_2} + \mathcal{N}, \tag{29}$$

where $\mathcal{N}$ is the contribution of the oscillatory modes. This mass is invariant under both $SL(2,Z)$'s of the torus acting on $\rho$ and $\tau$. Applying T-duality in $R_1$ direction,

$$R_1 \rightarrow \frac{1}{R_1} \quad \text{or equivalently} \quad \tau \leftrightarrow \rho,$$

we obtain the spectrum of the open string compactified on NCT,

$$M^2 = \frac{\rho_2}{|n+m\rho|^2}\frac{|r_1+q_1\tau|^2}{\tau_2} + \mathcal{N}, \tag{30}$$

Next we consider the D-string contribution. For this purpose we use the DBI action [9,20],

$$S_{D-string} = \frac{-1}{g_s}\int d^2\sigma \sqrt{det(\eta_{ab}+\mathcal{F}_{ab})}. \tag{31}$$

with

$$\eta_{ab} = \begin{pmatrix} 1-v^2 & 0 \\ 0 & 1 \end{pmatrix}, \tag{32}$$

$$\mathcal{F}_{ab} = \begin{pmatrix} 0 & Bv+F \\ Bv+F & 0 \end{pmatrix}. \tag{33}$$

This action leads to the masss pectrum

$$\alpha' M^2 = \frac{|n+m\tau|^2\rho_2}{\alpha' g_s^2 \tau_2} + \alpha' \frac{|r_2+\rho q_2|^2}{\rho_2\tau_2}. \tag{34}$$

Applying T-duality, we find,

$$\alpha' M^2_{membrane} = \frac{|n+m\rho|^2\tau_2}{\alpha' {g'_s}^2 \rho_2} + \alpha' \frac{|r_2+\tau q_2|^2}{\rho_2\tau_2}. \tag{35}$$

The $SL(2,Z)_N$ invariance, acting on $\rho$, is manifestly seen from the above equation. The open strings and the D-string form a *marginal* bound state, and the full BPS spectrum is the addition of the separate contributins,

$$\mathcal{M} = M_{membrane} + M_{open\ st.}. \tag{36}$$

$$\mathcal{M} = \sqrt{\frac{\tau_2}{\rho_2}}\frac{|n+m\rho|}{g'_s}(1+{g'_s}^2\frac{|r_2+q_2\tau|^2}{\tau_2}\frac{\rho_2}{|n+m\rho|^2})^{1/2} + \frac{|r_1+q_1\tau|}{|n+m\rho|}\sqrt{\frac{\rho_2}{\tau_2}}. \tag{37}$$

The above spectrum is manifestly $Sl(2,Z)_N$ invariant,in the notation of CDF. In the zero volume and $g_s \rightarrow 0$ limits,

$$l_s\mathcal{M} = \frac{|n-m\theta|}{g_s} + \frac{1}{2g_s}\frac{m^2V^2}{|n-m\theta|} + \frac{g_s}{2|n-m\theta|}\frac{|r_2+q_2\tau|^2}{\tau_2} + \frac{|r_1+q_1\tau|}{|n-m\theta|}\sqrt{\frac{V}{\tau_2}}. \tag{38}$$

The $SL(2,Z)_N$ symmetry generators are

$$\rho \to \rho+1 \quad , \quad \rho \to \frac{-1}{\rho}$$

which in the zero volume limit ($\rho_2 = 0$) become

$$\theta \to \theta+1 \quad , \quad \theta \to \frac{-1}{\theta} \tag{39}$$

Invariance of the mass spectrum, under $\theta \to \frac{-1}{\theta}$, implies that

$$g_s \to g_s' = g_s\theta^{-1} \tag{40}$$

Moreover the imaginary part of $\rho \to \frac{-1}{\rho}$, tells us that the volume of the torus in the zero volume limit, in the string theory units, transforms as:

$$V \to V' = V\theta^{-2} \tag{41}$$

Putting these relations together, and remembering the relation of 10 dimensional units and 11 dinemsional parameters, $l_p^3 = l_s^3 g_s$ and $l_s g_s = R$, and assuming $l_p$ invariance under $\theta$ transformations, we obtain:

$$\begin{aligned} R &\to R' = R\theta^{-2/3} \\ R_i &\to R_i' = R_i\theta^{-2/3} \\ l_s &\to l_s' = l_s\theta^{-1/3}. \end{aligned} \tag{42}$$

The above relations indicate an M-theoretic origin for the $SL(2,Z)_N$. This is the effect of considering the whole DBI action and not, only its second order terms[20]

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