On F-Theory/Heterotic Duality*

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Using local mirror symmetry in type IIA string compactifications on Calabi–Yau $n+1$ folds X_{n+1} we construct vector bundles on (possibly singular) elliptically fibered Calabi–Yau n-folds Z_n. As an application we find non-perturbative gauge enhancements of the heterotic string on singular Calabi–Yau manifolds and new non-perturbative dualities relating heterotic compactifications on different manifolds.

I. INTRODUCTION

The heterotic string compactified on a Calabi–Yau three-fold X_3 has been the phenomenologically most promising candidate amongst perturbatively defined string theories for quite some time [1]. In particular, compactifications with $(0,2)$ supersymmetry can easily lead to realistic gauge groups [2]. The definition of the theory involves the understanding of a suitable stable vector bundle V on X_3, which turns out to be a very difficult problem, however.

In this talk we discuss recent work on how F-theory and mirror symmetry can be used as a tool for understand heterotic compactifications [3]. In particular, we describe how both the choice of vector bundle V and the elliptically fibered Calabi-Yau Z_n are encoded in the F-theory vacuum X_{n+1} (modulo turning on fields which in the corresponding type IIA compactification on X_{n+1} are of RR type). From this one can read off both perturbative and non-perturbative gauge enhancements in the heterotic theory. For more details we refer the reader to [3].

II. HOLOMORPHIC BUNDLES, ELLIPTIC CALABI–YAU MANIFOLDS AND MIRROR SYMMETRY

Recall that the basic duality is between F-theory on elliptically fibered K3 and the heterotic string on T^2 in eight dimensions [4,5]. Lower dimensional dualities are obtained by "fibering the eight-dimensional duality" with the result that F-theory on an elliptically and K3 fibered n+1-dimensional Calabi–Yau manifold X_{n+1} is dual to the heterotic string on an elliptically fibered n-dimensional Calabi–Yau Z_n [8].

We will take a rather different path to derive this duality, which as a bonus will lead to interesting non-perturbative results. Consider a type IIA compactification on K3$\times T^2$ where the K3 is elliptically fibered and has a singularity of type H. Part of the moduli space $\mathcal{M}_{IIA}$ is identified with the moduli space $\mathcal{M}_{T^2}$ of Wilson lines on T^2. The R-symmetry of the $N=4$ supersymmetry of this compactification provides identifications in $\mathcal{M}_{IIA}$ which in particular relate Kähler deformations of the singularity H in the elliptic fibration of the K3 to $\mathcal{M}_{T^2}$ [19]. Application of local mirror symmetry maps this description of $\mathcal{M}_{T^2}$ in terms of Kähler moduli to a description of $\mathcal{M}_{T^2}$ in terms of complex deformations of a local mirror geometry[1] $\mathcal{W}_2$. In particular, $\mathcal{W}_2$ gives a concrete description of the elliptic curve T^2 and a flat H bundle on it, where H is the type of the original singularity we started with. After having understood the relation between the moduli space of elliptically fibered K3 and flat bundles on E, we can discard the T^2 in the above discussion and consider the six-dimensional type IIA compactification on the K3, M_2. Thus, we start with an elliptically fibered K3 manifold $M_2(H)$ with a singularity of type H in the elliptic fibration. Applying mirror symmetry to $M_2(H)$ we obtain a mirror K3, $W_2(H)$, with the roles of complex and Kähler deformations exchanged. In the local mirror limit we obtain a complex geometry $\mathcal{W}_2$ whose deformations describe the H bundle on E_H. Fibering

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[1] Here and in the following a subscript denotes the complex dimension of a geometry.

the local geometry $\mathcal{W}_2$ over a complex base manifold B we obtain an H bundle on the elliptically fibered manifold $Z \to B$ with fibers E_b, $b \in B$.

Up to now we have obtained the moduli space of H bundles on Z as complex deformations of $\mathcal{W}_2$, or more precisely the piece of it that is described by the data which determine the restriction of the bundle V to the fibers E_b. In addition we have Kähler deformations describing another flat bundle on T^2. This is also the classical moduli space of the heterotic string on $Z \times T^2$ with a special vector bundle that splits into a bundle on Z and Wilson lines on T^2. In the small fiber limit this is reduced to the heterotic string on Z. The statement that the full moduli spaces are equivalent is the conjectured duality between the heterotic string and the corresponding type IIA/F-theory. In six dimensions, we can interpret the complex and Kähler deformations of the elliptically fibered manifold W_2 as two sets of Wilson lines W_I and W_{II} with structure groups H_I and H_{II} on a $T_I^2 \times T_{II}^2$ compactification of the heterotic string, respectively. Specifically, the K3 W_2 provides a Kähler resolution of an H_I singularity, whereas the complex deformations encode the resolution of an H_{II} singularity. The decompactification limit of T_I^2 switches off the Wilson lines in H_I and restores an H_I gauge symmetry. It corresponds to the small fiber limit of W_2 which blows down the Kähler resolution of the H_I singularity in the elliptic fibration. Thus blowing down the generic fiber blows down the spheres of the H_I singularity leading to a gauge symmetry enhancement in the type IIA theory.

The classical limit of the moduli space $\mathcal{M}_E(H)$ of H bundles on E, as represented by a Kähler deformation of an H singularity in an elliptic fibration of a K3, M_2, corresponds to a limit where we consider only the local deformations of the H singularity and switch off the coupling to the global geometry. However, we are really interested in taking the same limit for the *complex* deformations of the mirror $W_2(H)$. To be specific consider the case of K3 manifolds dual to the $E_8 \times E_8$ string. In this case the K3 manifold $M_2(H)$ has generically two singularities at the points $z = 0$ and $z = \infty$ corresponding to the eight-dimensional gauge group $H = H_1 \times H_2$ in the two E_8 factors. We represent $M_2(H)$ and $W_2(H)$ as hypersurfaces in a toric variety. The mirror manifold $W_2(H)$ is described as a hypersurface, given as the zero locus of a polynomial $p_{W_2(H)}$ in the toric embedding space. We assert that for the class of K3's dual to $E_8 \times E_8$ string, the polynomial $p_{W_2(H)}$ takes the general form

$$\begin{aligned} p_{W_2(H)} &= p_0 + p_+ + p_- \ , \\ p_0 &= y^2 + x^3 + \tilde{z}^6 + \mu x y \tilde{z} \ , \\ p_\pm &= \textstyle\sum_{i=1}^k v^{\pm i} p_\pm^i \ , \end{aligned} \tag{1}$$

where $y, x, \tilde{z}, v$ are specially chosen coordinates on the embedding space; in particular v is a coordinate on the base $\mathbf{P}^1$. Moreover $p_\pm^i$ are polynomials in $y, x, \tilde{z}$ of homogeneous degree with respect to the scaling action $(y, x, \tilde{z}) \to (\lambda^3 y, \lambda^2 x, \lambda \tilde{z})$, and μ is a complex parameter related to the complex structure of the elliptic curve $\hat{E}_H : p_0 = 0$.

We claim that the local mirror corresponds to take a limit in the complex parameters such that

$$p_-^i \to \epsilon^i p_-^i \ , \tag{2}$$

with $\epsilon \to 0$. The local mirror geometry $\mathcal{W}$ is then given by

$$p_{\mathcal{W}} = p_0 + p_+ = 0. \tag{3}$$

The complex deformation of the geometry $p_{\mathcal{W}} = 0$ describes the moduli space of a flat bundle V compactified on the elliptic curve E_H defined by the hypersurface $v = 0$, $\hat{E}_H \, : \, p_0 = 0$ [2]. The polynomial p_+ contains the information about a bundle V_+ on E_H. E.g. for $H = SU(N)$, N even, we will have

$$p_+ = v \, (\tilde{z}^N + \tilde{z}^{N-2} x + \tilde{z}^{N-3} y + \ldots + x^{N/2}) \ . \tag{4}$$

[2] More precisely $\hat{E}_H$ is the dual of the torus E_H.

We can now integrate out v and obtain a geometry defined by the two equations $p_0 = 0 \cap p_+ = 0$. This intersection gives N points on $\hat{E}_H$, which are interpreted as the values of the Wilson lines in the Cartan algebra of $SU(N)$. These data specify uniquely the $SU(N)$ bundle V_+ [6].

Via duality, we will interpret this bundle as a bundle in the first E_8 factor of the heterotic string. Since the original K3 had two singularities, the limit (2) must already include a choice of neighborhood. To describe the neighbourhood of the second singularity, we simply rescale the variable $v \to v\epsilon$ with the result that now the perturbations in p_+^i scale as ϵ^i while those in p_-^i are constant. The corresponding bundle V_- can be interpreted as the bundle in the second E_8 factor of the dual heterotic string.

The above construction can be generalized to lower-dimensional dual pairs of F-theory on Calabi–Yau W_{n+1} and the heterotic string on Calabi–Yau Z_n by an application of the adiabatic argument [9]. The geometry $W_2(H)$ describes an H bundle over E_H in the local limit. To obtain the description of an H bundle over an elliptically fibered Calabi–Yau Z_n we can fiber the geometry $W_2(H)$ over an $n-1$ dimensional base B_{n-1} to obtain a Calabi–Yau W_{n+1}. In the local limit we now get an n dimensional geometry $\mathcal{W}$ defined as in (1), but with the polynomials $p_\pm$ being functions of the coordinates of the base B_{n-1} (or rather sections of line bundles on B_{n-1}). Similarly the bundle is now defined on the projection to $p_0 = 0$, which gives an n dimensional Calabi–Yau $\hat{Z}_n$. We can identify $\hat{Z}_n$ with the dual[3] $\hat{Z}_H$ of a dual heterotic compactification manifold Z_H. Finally, we note that the local limit is taken only in the fiber $W_2(H)$ (we choose to concentrate on the point with the singularity in the K3 fiber), but we retain the global structure of the elliptic fibration over the base B_{n-1}.

To obtain a description of holomorphic bundles on elliptic (Calabi–Yau) manifolds we need a toric description of a fibration of the local mirror geometry $\mathcal{W}_2$ above a base manifold B_{n-1}. For our discussion we only need to know the following results. In [10] it was shown that a toric manifold X_n admits a fibration with Calabi–Yau fibers Y_k, if its polyhedron $\Delta^*_{X_n}$ contains the polyhedron of the fiber Y_k as a hypersurface of codimension $n-k$. Thus a fibration of $\mathcal{W}_2 \subset W_2(H)$ over an $n-1$ dimensional base B_{n-1} is described by an $n+2$ dimensional polyhedron $\Delta^*_{W_{n+1}}$ that contains $\Delta^*_{\mathcal{W}_2}$ as a *hyperplane*. Furthermore, $Z_n : p_0 = 0$ defines an n dimensional Calabi–Yau manifold. The holomorphic H bundle is defined on Z_n and we are free to interpret this data as a classical heterotic vacuum. Since the "heterotic manifold" Z_n is defined by $p_0 = 0$ which contains the monomials with zero power of v, the heterotic manifold Z_n corresponds to a *projection* $f : W_{n+1} \to Z_n$.

III. SIX-DIMENSIONAL HETEROTIC $N = 1$ VACUA

Let us proceed with six-dimensional dual pairs, that is F-theory on Calabi–Yau three-fold W_3 versus the heterotic string on K3 [11–13] [4]. For illustration we will mostly be concerned with the $E_8 \times E_8$ case, unless otherwise stated. The moduli space of the type IIA compactification on W_3 has now two sectors, the moduli space $\mathcal{M}_{HM}$ parameterized by the hypermultiplets and the vector multiplet moduli space $\mathcal{M}_{VM}$. These spaces are in general decoupled due to the constraints of $N = 2$ supersymmetry up to subtleties explained e.g. in [14].

To specify the theory we have to choose two bundles V_1, V_2 in the two E_8 factors and the integer n which specifies the fibration of the K3 fiber M_2 over a further $\mathbf{P}^1$. The bundle (V_1, V_2) determines the K3 fiber M_2, while n corresponds also to the way the total instanton number $k = 24$ is divided between the two E_8 factors: $k_1 = 12 + n$, $k_2 = 12 - n$ [8]. We take the structure group of V_2 to be trivial and concentrate on the first E_8 factor. The instanton number k_1 is encoded in the fibration of $M_2(H_1)$ over the base $\mathbf{P}^1$ with coordinates (s, t) and corresponds to the choice of n for the Hirzebruch surface $\mathbf{F}_n$ [8].

[3] With duality understood as the replacement of the elliptic fiber E_H by the dual $\hat{E}_H$ parametrizing the Jacobian of E_H. For simplification we will drop the hat on Z_n in the following.

[4] In particular the result of local mirror symmetry is technically closely related to the stable degenerations of Calabi–Yau manifolds introduced in [6] and discussed further in [11].

As an example consider the case of an $SU(N)$ bundle, where we choose the Calabi-Yau three-fold W_3 to have a K3-fiber $W_2(A_N)$ describing the mirror of the $H_1 = A_{N-1}$ singularity given by $M_2(H_1)$ above. In the local limit we obtain the three-fold geometry $\mathcal{W}_3$:

$$
\begin{aligned}
p_{\Delta_4^*}|_{\rm local} &= p_0 + p_+ \ , \\
p_0 &= y^2 + x^3 + \tilde{z}^6 f_{12} + y\tilde{z}^3 h_6 + x\tilde{z}^4 h_8 + x^2\tilde{z}^2 h_4 + yx\tilde{z}st \ , \\
p_+ &= v\,(\tilde{z}^N f_{k_1} + x\tilde{z}^{N-2} f_{k_1-4} + \ldots + \left\{ {x^{N/2} f_{k_1-2N} \atop yx^{\frac{N-3}{2}} f_{k_1-2N}} \right\}).
\end{aligned} \tag{5}
$$

Here f_l is a generic polynomial of homogeneous degree l in the variables (s,t) while h_l is of the restricted form $h_l = s^l + \alpha_l t^l$. The interpretation of the three complex dimensional geometry $\mathcal{W}_3$ is very similar to the situation we encountered before: $v = 0$ projects onto the K3 surface $Z_2 : p_0 = 0$. This is the K3 surface (dual to the manifold) on which the heterotic string is compactified. Integrating out the linear variable v we obtain a one-dimensional geometry, the intersection $p_0 = 0 \ \cap p_+ = 0$ which describes a curve C in Z_2. C is the spectral curve which determines the $SU(N)$ bundle on Z_2 as in [6,7].

The number of parameters of p_+ is $Nk_1 - N^2 + 2$. Discarding one parameter which can be absorbed in an overall rescaling this agrees with the dimension of the moduli space of A_N bundles of instanton number k on K3 as determined by the index formula

$$
\dim \mathcal{M}(H) = c_2(H)\, k - dim(H) \ , \tag{6}
$$

which applies for simple H and large enough k [5]. The above analysis can be repeated for any ADE-singularity, the results of which can be found in [19,3].

So far we have only described heterotic vacua where the manifold Z_H is smooth and the vector bundle H is generic, leading to perturbative gauge symmetries with group G, the commutant of H in original ten-dimensional gauge group G_0. Using F-theory knowledge of how to engineer non-perturbative gauge symmetries we obtain a blow-up of W_3 corresponding to a non-perturbative gauge symmetry G_{np} [8,22]. This is done by wrapping a seven-brane on the fiber $\mathbf{P}^1$ of the base $B_F = \mathbf{F}_n$ of the elliptic fibration $\pi'_F : \ W \to B_F$. The geometry $\mathcal{W}'_3$ obtained from the blown up three-fold defines the heterotic data in the same way as for the case of smooth bundles and smooth Z_2. In particular, we get the following result: the $E_8 \times E_8$ heterotic string compactified on an elliptically fibered K3 with a singularity of type $\mathcal{G}$ at a point $s = 0$ and a special gauge background $\hat{V}$, acquires a non-perturbative gauge symmetry $G_{np} \supset \mathcal{G}$ if the restriction $\hat{V}_{|E_H}$ to the fiber E_H at $s = 0$ is sufficiently trivial [6].

The verification of the above claim is very simple using the fact that the bundle is defined on the Calabi–Yau $Z_2 : p_0 = 0$. In terms of the generalized Weierstrass form the conditions for a singularity of type $\mathcal{G}$ have been analyzed using the Tate's formalism in [22]. The singularity at a point $s = 0$ is determined by the powers of vanishing of the coefficients in (5). Since the polynomial p_0 consists of a subset of the polynomials in (5), the coefficients f_n, h_n of the generalized Weierstrass form of Z_2 fulfil the same singularity condition as p, so Z_2 has a $\mathcal{G}$ singularity.

We will now turn to non-perturbative equivalences which occurs in a certain class of six-dimensional heterotic theories with large non-perturbative groups. This is implemented using the toric map $f : \ W_3 \to Z_2$; recall that the elliptically fibered Calabi-Yau W_3 has a K3 fiber W_2 given by a hyperplane $\mathcal{H} = \Delta^\star_{W_2}$ in $\Delta^\star_{W_3}$ and the heterotic K3 Z_2 appears as the projection $f : \ \Delta^\star_{W_3} \to \Delta^\star_{Z_2}$.

The key feature is when the projection f results in a hyperplane $\mathcal{H}'$, that is W_3 admits at the same time a Z_2 fibration. The two K3 fibrations, W_2, Z_2 imply that we have two different perturbatively defined heterotic theories

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[5] Note that for large N we have to consider non-compact Calabi–Yau geometries as in [19].

[6] The enhancement of gauge symmetry requires in addition appropriate values for the non-geometric moduli associated to a line bundle $\mathcal{L}$ on W_3 [6]. Furthermore, the non-perturbative gauge group can be larger than $\mathcal{G}$, if additional restrictions on the behavior of $\hat{V}$ in a neighbourhood of the singularity are imposed.

in four dimensions, which are non-perturbatively equivalent[7]. Moreover, if the two K3 manifolds defined by the hyperplanes $\mathcal{H}$ and $\mathcal{H}'$ share the same elliptic fiber, we can take the F-theory limit without interfering with the equivalence[8]. In this way we obtain two six-dimensional heterotic compactifications on Z_2 and W_2 which are non-perturbatively equivalent. These manifolds can be at rather different moduli, one being highly singular while the other being smooth, as we will see in the following example.

Let us start with the simplest case corresponding to a heterotic theory with 24 small E_8 instantons located at two points in a smooth K3 Z_2. The perturbative gauge group is $E_8 \times E_8$. The K3 fiber of the three-fold W_3 is therefore the K3 W_2 with an $E_8 \times E_8$ singularity. It is easy to show that the Calabi–Yau manifolds W_3 associated to $\Delta^*_{W_3}$ give indeed the correct physics. Firstly, the hodge numbers are $h^{1,1} = 43(0)$, $h^{1,2} = 43(22)$, where the number in parentheses denotes the number $\delta h^{1,1}$ ($\delta h^{1,2}$) of so-called non-toric (non-polynomial) deformations, which are not available in the toric model. The $n_T + n_V = h^{1,1} - 2$ vector and tensor multiplets are associated to the 16 vector multiplets of $E_8 \times E_8$, 24 tensor multiplets from the 24 small E_8 instantons and the heterotic coupling (2 Kähler classes corresponding to the volume of the elliptic fiber and the volume of the base do not contribute to the vector and tensor multiplets [8]). The $n_H = h^{2,1} + 1$ hypermultiplets arise from the 20 moduli from K3 and 2 moduli for the two positions of the two groups of fivebranes. The 22 missing complex structure moduli correspond naturally to the fact that we have fixed 22 of 24 positions of the small instantons in the K3 Z_2.

There is a second elliptic fibration of the K3 fiber W_2 corresponding to the gauge group $SO(32)$ [16]. Instead of $n_T = 24$ extra tensors we have in this case an $Sp(a+b) \times Sp(24-a-b)$ gauge group from two groups of coincident $SO(32)$ five branes [17] with matter in the $(2k, 32) \oplus (1,1) \oplus (2k^2 - k - 1, 1)$ of each $Sp(k) \times SO(32)$ factor.

These are the first two interpretations of F-theory compactification on W_3. However, we have to shrink different elliptic fibers E_1 and E_2 in the F-theory limit [3], so these theories are disconnected in the small fiber limit in six dimensions and become equivalent only in five dimensions by T-duality [18].

More interestingly, since the base K3 Z_2, corresponding to the heterotic compactification manifold, does not only appear as a projection but as a hyperplane in $\Delta^*_{W_3}$, there is a second K3 fibration with fiber Z_2 which is itself elliptically fibered *with the same elliptic fiber* E_1 as the K3 fiber of the original K3 fibration. Therefore we obtain a theory in six dimensions which is non-perturbatively equivalent to the heterotic string with 24 small instantons on on smooth K3.

We interpret now the smooth K3, Z_2, as the fiber K3. Due to the absence of a singularity, the perturbative gauge group must be trivial and therefore the bundle V_0 has structure group $E_8 \times E_8$ on the generic elliptic fiber. On the other hand, in the new K3 fibration, $W_2^{E_8 \times E_8}$ has become the base K3. The heterotic compactification manifold has therefore an $E_8 \times E_8$ singularity. The perturbative $E_8 \times E_8$ gauge symmetry of the compactification with small instantons is produced in the dual theory purely by non-perturbative effects related to the singularities of the manifold and the bundle. For $a = b = 12$ we have therefore the following duality:

(†) The $E_8 \times E_8$ heterotic string compactified on a smooth K3 with two groups of 12 small instantons is non-perturbatively equivalent to compactifying on a K3 $p_0 = 0$ with $E_8 \times E_8$ singularity with gauge bundle V_0. Here

$$p_0 = y^2 + x^3 + yx\tilde{z}st + \tilde{z}^6(s^7t^5 + s^6t^6 + s^5t^7) \ . \tag{7}$$

The $E_8 \times E_8$ bundle V_0 is specified by a geometry $\mathcal{W}$ of the special form

$$p_+ = \tilde{v}(z'^5 + yx) + \tilde{v}^2(z'^4 + x^2) + \tilde{v}^3(z'^3 + y) + \tilde{v}^4(z'^2 + x) + \tilde{v}^5 z' + \tilde{v}^6 \ . \tag{8}$$

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[7] The quite reverse situation is known to occur, in which two heterotic theories have the same perturbative spectrum while non-perturbatively they are different [15].

[8] To be precise, in order for the F-theory limit to work, we have to require that not only $\mathcal{H}'$ but also the hyperplane $\mathcal{H}$ coincides with a projection.

with $\tilde{v} = vst$, $z' = \tilde{z}st$ and a similar polynomial for p_- for the other E_8 factor.
One can show that similar results hold for singular K3 manifolds [3].

IV. N=1 SUPERSYMMETRIC VACUA IN FOUR DIMENSIONS

To get a four-dimensional theory by considering the F-theory compactification associated to the local mirror limit of the type IIA geometry we fiber the local geometries $\mathcal{W}_2$ over a two complex dimensional base B_2. B_2 will also be the base of the elliptic fibration $\pi_H: Z_3 \to B_2$ of the Calabi–Yau three-fold Z_3 on which the bundle V is defined. A toric representation can be given for the cases $B_2 = \mathbf{P}^2$ or $\mathbf{F}_n$, or a series of blow ups thereof.

Rather than describing the details of the construction of the local geometry, which is done in complete analogy with the analysis of the previous sections, we will directly turn to interesting non-perturbative dualities. Let $\Delta_5^\star$ describe the polyhedron associated to a Calabi–Yau manifold W_4. There is a codimension two hyperplane $\mathcal{H} = \Delta_3^\star$ associated to an elliptically fibered K3, W_2. The elliptic fiber is described by a hyperplane $\Delta_2^\star(E)$ in $\Delta_3^\star$. The W_2 fibration of W_4 describes H bundles on a Calabi–Yau three-fold Z_3 given by a projection in the direction of the section of W_2. In our conventions the hyperplane is given by $x_1 = x_2 = 0$ and the projection is in the third direction. A non-perturbative dual in four dimensions exists if *i*) there is a second codimension two hyperplane $\mathcal{H}'$ that describes an elliptically fibered K3 manifold W_2' with a section and the same elliptic fiber E as W_2: $\mathcal{H} \cap \mathcal{H}' = \Delta_2^\star(E)$; *ii*) the projection in the direction of the section of W_2' results in a reflexive polyhedron for a Calabi–Yau three-fold Z_3'. We have:

($\dagger\dagger\dagger$) *Let the heterotic string be compactified on a Calabi–Yau three-fold with $\mathcal{G}$ singularity and with a certain gauge background with structure group H such that the toric data $\Delta_5^\star$ fulfil conditions i) and ii). Then there exists a non-perturbatively equivalent compactification on a Calabi–Yau manifold with G singularity and with a specific gauge background with structure group $\mathcal{H}$.*

Again H ($\mathcal{H}$) is the commutant of G ($\mathcal{G}$) in $E_8 \times E_8$.

Because of the geometry it is easy to see that we in fact can have up to three different K3-fibrations. Correspondingly we have a triality of non-perturbative equivalences. It would be interesting to see how this arises in the heterotic string.

V. CONCLUSIONS

We have seen how mirror symmetry can be used to define vector bundles on Calabi–Yau n-folds Z_n and implies F-theory/heterotic duality at the classical level. In particular the construction allows for a very systematic identification of a dual pair realized in toric geometry, consisting of a Calabi–Yau $n+1$-fold W_{n+1} for F-theory compactification and a Calabi–Yau n-fold Z_n together with a family of vector bundles on it defining a heterotic theory. However, it is important to remember that the non-perturbative heterotic physics has been derived from F-theory. We would like to find a more direct understanding in terms of the heterotic string and five-brane without relying on duality.

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