Wormholes and Flux Tubes in Kaluza-Klein Theory

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In this work spherically symmetric solutions to 5D Kaluza-Klein theory, with "electric" and/or "magnetic" fields are examined. Different relative strengths of the "electric" and "magnetic" charges of the solutions are studied by varying certain parameters in our metric ansatz. As the strengths of these two charges are varied the resultant spacetime exhibits an interesting "evolution".

I. INTRODUCTION

In this work we investigate a class of spherically symmetric metrics in multidimensional (MD) gravity. The metric ansatz which is used has off diagonal elements which leads to these solutions having "electric" and/or "magnetic" charges. The solutions examined here are either MD wormholes or infinite/finite flux tubes. It is found that the type of solution obtained depends crucially on the relative magnitudes of these charges and thus on the form of the off-diagonal metric components. Usually in the discussion of wormhole or blackhole solutions such off-diagonal elements are not considered, (see for, example, [1]- [5]).

The off-diagonal components of the MD metric play the role of gauge fields (U(1), SU(2) or SU(3) gauge fields), and a scalar field $\phi(x^\mu)$ which is connected with the linear size of the extra dimension. These geometrical fields can act as the source of the exotic matter necessary for the formation of the wormhole's mouth. Such solutions were obtained in Refs. [6] [7] [8] [9]. These works indicate that the exotic matter necessary for the formation of the WH can appear in ***vacuum multidimensional gravity*** from the off-diagonal elements of the metric (the gauge fields) and from the G_{55} component of the metric (the scalar field), rather than coming from some externally given exotic matter.

In Refs. [8], [9] a MD metric with only "electric" fields was investigated. In Ref [10] a MD metric with "magnetic" field = "electrical" field was investigated. In this paper we investigate the consequence of having both "electric" and "magnetic" Kaluza-Klein fields of varying relative strengths. We will consider 5D Kaluza-Klein theory as gravity on the principal bundle with U(1) fibre and 4D space as the base of this bundle [9].

II. THE FIELD EQUATIONS

For our spherically symmetric 5D metric we take

$$\begin{aligned} ds^2 = & e^{2\nu(r)}dt^2 - r_0^2 e^{2\psi(r)-2\nu(r)}\left[d\chi + \omega(r)dt + n\cos\theta d\varphi\right]^2 \\ & - dr^2 - a(r)(d\theta^2 + \sin^2\theta d\varphi^2), \end{aligned} \tag{1}$$

where χ is the 5^{th} extra coordinate; r, θ, φ are $3D$ spherical-polar coordinates; n is integer; $r \in \{-R_0, +R_0\}$ (R_0 may be equal to ∞). We require that all functions $\nu(r), \psi(r)$ and $a(r)$ be even functions of r and hence $\nu'(0) = \psi'(0) = a'(0) = 0$. The ansatz function $\omega(r)$ is the t-component of the electromagnetic potential and $(n\cos\theta)$ is the φ-component. Thus we have radial Kaluza-Klein "electrical" and "magnetic" fields.

Substituting this ansatz into the 5D Einstein vacuum equations gives [11] the following set of coupled, non-linear differential equations

$$\nu'' + \nu'\psi' + \frac{a'\nu'}{a} - \frac{1}{2}r_0^2\omega'^2 e^{2\psi-4\nu} = 0, \tag{2}$$

$$\omega'' - 4\nu'\omega' + 3\omega'\psi' + \frac{a'\omega'}{a} = 0, \tag{3}$$

$$\frac{a''}{a} + \frac{a'\psi'}{a} - \frac{2}{a} + \frac{Q^2}{a^2} e^{2\psi - 2\nu} = 0, \tag{4}$$

$$\psi'' + \psi'^2 + \frac{a'\psi'}{a} - \frac{Q^2}{2a^2} e^{2\psi - 2\nu} = 0, \tag{5}$$

$$\nu'^2 - \nu'\psi' - \frac{a'\psi'}{a} + \frac{1}{a} - \frac{a'^2}{4a^2} - \frac{1}{4} r_0^2 \omega'^2 e^{2\psi - 4\nu} - \frac{Q^2}{4a^2} e^{2\psi - 2\nu} = 0 \tag{6}$$

The Kaluza-Klein "magnetic" charge is $Q = nr_0$. The Kaluza-Klein "electrical" field can be defined by multiplying Eq. (3) by $4\pi r_0$ and rewriting it as

$$\left(r_0 \omega' e^{3\psi - 4\nu} 4\pi a\right)' = 0. \tag{7}$$

This can be compared with the normal 4D Gauss's Law

$$(E_{4D} S)' = 0, \tag{8}$$

where E_{4D} is 4D electrical field and $S = 4\pi r^2$ is the area of 2-sphere S^2. Eqs. (2) - (6) are five equations for determining the four ansatz functions (ν, ψ, a, ω). The first four equations (Eqs. (2 - 5)) are dynamical equations which determine the ansatz functions, while the last equation (Eq. (6)) contains no new dynamical information not contained in the first four equations, but gives some initial conditions related to solving this system of equations. For the metric given in Eq. (1) r^2 is replaced by $a(r)$ and the surface area is given by $S = 4\pi a(r)$. Comparing Eq. (7) with Eq. (8) we can identify the 5D Kaluza-Klein "electric" field as

$$E_{KK} = r_0 \omega' e^{3\psi - 4\nu} \tag{9}$$

If we integrate Eq. (7) once and let the integration constant be $4\pi q$, then from Eq. (9) we find that $E_{KK} = q/a(r)$ where q can be taken as the Kaluza-Klein "electric" charge. Finally for the system of equations given in Eqs. (2) - (6) we will consider solutions with the boundary conditions $a(0) = 1, \psi(0) = \nu(0) = 0$ (for numerical calculations we introduced dimensionless function $a(r) \to a(r)/a(0)$ and $x = r/a(0)$). Using these boundary conditions in Eq. (6) and also in Eq. (9) (which gives $r_0 w'(0) = q$) gives the following relationship between the Kaluza-Klein "electric" and "magnetic" charges

$$1 = \frac{q^2 + Q^2}{4a(0)} \tag{10}$$

From Eq. (10) it is seen that the charges can be parameterized as $q = 2\sqrt{a(0)}\sin\alpha$ and $Q = 2\sqrt{a(0)}\cos\alpha$.

We will examine the following cases:

A) $Q = 0$ or $H_{KK} = 0$, "magnetic" field is zero.

B) $q = 0$ or $E_{KK} = 0$, "electrical" field is zero.

C) $H_{KK} = E_{KK}$, "electrical" field equal to "magnetic" field.

D) $H_{KK} < E_{KK}$, "magnetic" field less than "electrical".

E) $H_{KK} > E_{KK}$, "electrical" field less than "magnetic".

A. Switched off "magnetic" field.

In this case we have the following solution [6] [8]:

$$
\begin{aligned}
a &= r_0^2 + r^2, && (11)\\
e^{2\nu} &= \frac{2r_0}{q}\frac{r_0^2+r^2}{r_0^2-r^2}, && (12)\\
\psi &= 0 && (13)\\
\omega &= \frac{4r_0}{q}\frac{r}{r_0^2-r^2}. && (14)
\end{aligned}
$$

This WH-like spacetime has a nonasymptotically flat metric, bounded by two surfaces at $r = \pm r_0$ where the reduction from 5D to 4D spacetime breaks down. As r moves away from 0 the cross-sectional size of the throat, $a(r)$, increases.

A connection can be made between the present solution and Wheeler's old proposal of electric charge as a wormhole filled with electric flux that flows from one mouth to the other – the "charge without charge" model of electric charge. In a recent work [12] a model of electric charge along these lines was proposed where electric charge is modeled as a kind of composite WH with a quantum mechanical splitting off of the 5^{th} dimension. The 5D WH-like solution of Eqs. (11-14) have two Reissner-Nordström black holes attached to it on the surfaces at $\pm r_0$. By considering 4D electrogravity as a 5D Kaluza-Klein theory in the initial Kaluza formulation with $G_{55} = 1$ we can join the 5D and Reissner-Nordström solutions at the $r = \pm r_0$ surfaces base to base and fibre to fibre.

B. Switched off "electrical" field

In this case we will simplify by taking $\nu = 0$ in addition to $\omega = 0$ so that the equations reduce to

$$
\begin{aligned}
&\frac{y''}{y} + \frac{y'a'}{ya} - \frac{Q^2y^2}{2a^2} = 0, && (15)\\
&\frac{a''}{a} + \frac{y'a'}{ya} - \frac{2}{a} + \frac{Q^2y^2}{a^2} = 0, && (16)\\
&\frac{a'y'}{ay} - \frac{1}{a} + \frac{a'^2}{4a^2} + \frac{Q^2y^2}{4a^2} = 0 && (17)
\end{aligned}
$$

where $y(r) = \exp(\psi(r))$. These are three equations for two ansatz functions, $\psi(r), a(r)$. The last equation, Eq. (17), simply repeats information that is already contained in the first two equations. We solved the system of equations (15) - (16) numerically, using the *Mathematica* package, with the following initial conditions: $a(0) = a_0 = 1$, $a'(0) = 0$, $y(0) = 1, y'(0) = 0$, (where we are using the dimensionless quantities $x = r/a_0$ and $a \to a/a_0$). These conditions and $\alpha = 0$ fix the dimensionless "magnetic" charge as $Q = 2$. The detailed results of the numerical calculations for $a(r)$ and $y(r)$ are given in Ref. [11]. The general shape of the cross-section function, $a(r)$ of this solution can be seen in the last picture of Fig. 1. Also from this picture one can see two singularities at $x = \pm x_0$. We interpret these singularities as the location of two magnetic charges ($\pm Q$) with flux lines of Kaluza-Klein "magnetic" field going from $+Q$ to $-Q$. Near these singularities the ansatz functions have the following asymptotic behaviour:

$$
\begin{aligned}
y(r) &\approx \frac{y_\infty}{(r_0-r)^{1/3}}, && (18)\\
a(r) &\approx a_\infty (r_0-r)^{2/3}, && (19)\\
\frac{Qy_\infty}{a_\infty} &= \frac{2}{3}. && (20)
\end{aligned}
$$

The time part of the metric appears not to be influenced by the strong gravitational field since $G_{tt}(r) = \exp(2\nu(r)) = 1$. This result is similar to what was found in Ref. [13] [14] where "magnetic" Kaluza-Klein components of the metric were considered. One difference between the present solutions and the monopole solutions of Ref. [13] [14], is that the monopole solutions had only coordinate singularities, while $r = \pm r_0$ are real singularities for the present solution. This can be seen by calculating the invariant $R_{AB}R^{AB}$ and using the asymptotic form for $y(r), a(r)$ given in Eqs. (18) - (20). In Ref. [11] it was found that

$$R_{AB}R^{AB} \propto \frac{1}{(r_0 - r)^2} \tag{21}$$

Finally it can be shown that this spacetime has a finite volume V, by calculating $V = \int \sqrt{-G} d^5 v$. Near the singularities $r = \pm r_0$ we have:

$$\sqrt{-G} = \sqrt{-det(G_{AB})} = r_0 a(r) \exp\left(\psi(r)\right) \sin\theta \approx (r_0 - r)^{1/3} \to 0 \tag{22}$$

The form of this solution is suggestive of the color field flux tubes which are conjectured to form between two quarks in some pictures of confinement (see for example pg. 548 of Ref. [15]).

C. "Magnetic" field equal to "electrical" field

In this case $Q = q$ and an exact solution can be given [10]:

$$a = \frac{q^2}{2} = const, \tag{23}$$

$$e^{\psi} = e^{\nu} = \cosh \frac{r\sqrt{2}}{q}, \tag{24}$$

$$\omega = \frac{\sqrt{2}}{r_0} \sinh \frac{r\sqrt{2}}{q} \tag{25}$$

Using this solution and Eq. (9) we find that the Kaluza-Klein "electrical" field is

$$E_{KK} = \frac{q}{a} = \frac{2}{q} = const. \tag{26}$$

A similar magnetic flux tube-like solution was discussed in Ref. [16]. The Kaluza-Klein "magnetic" field associated with this solution is [10]

$$H_{KK} = \frac{r_0 n}{a} = \frac{Q}{a} = const. \tag{27}$$

Thus, this solution is ***an infinite flux tube*** with constant Kaluza-Klein "electrical" and "magnetic" fields. The direction of both the "electric" and "magnetic" fields is along the $\hat{\mathbf{r}}$ direction (*i.e.* along the axis of the flux tube). The sources of these Kaluza-Klein fields (5D "electrical" and "magnetic" charges) are located at $\pm\infty$. This feature leads us to consider this solution as a kind of 5D "electrical" and "magnetic" dipole.

D. Intermediate cases

We consider two different cases: $E_{KK} > H_{KK}$ (or $q > Q$) and $E_{KK} < H_{KK}$ (or $q < Q$). The initial conditions for both cases are taken as : $\psi(0) = \nu(0) = 0$, $\psi'(0) = \nu'(0) = 0$ and $a(0) = 1, a'(0) = 0$. These initial conditions along with a choice of α determine the magnitude of the charges q, Q. As in the "magnetic" case we solved the system of equations numerically [11].

1. $E_{KK} > H_{KK}$

As the "magnetic" field increases from 0 to $H_{KK} = E_{KK}$ we found the following behaviour: First, compared to the WH-like solution of the pure "electric" case, the longitudinal distance between the surfaces $\pm r_0$ is stretched as the magnetic field strength increases; second, the cross-sectional size of the solution, represented by the function $a(r)$ did not increase as rapidly as $r \to \pm r_0$. In the limit where the "magnetic" field equals the "electrical" field, $H_{KK} = E_{KK}$, the longitudinal length of the solution goes to ∞ and the cross-sectional size became a constant.

2. $E_{KK} < H_{KK}$

In this case the "electrical" field is taken as decreasing from the $E_{KK} = H_{KK}$ down to $E_{KK} = 0$. As the "magnetic" field strength increases relative to the "electric" field strength we notice the following evolution of the solution : the infinite flux tube of the equal field case turns into a finite flux tube when E_{KK} drops below H_{KK}. Also the cross-sectional size of this case has a maximum at $r = 0$ and decreases as $r \to \pm r_0$ where the singularities occur. We take these singularities as the locations of the "electric" / "magnetic" charges. Between the charges there is a flux tube of Kaluza-Klein "electric" and "magnetic" fields. The longitudinal size of this flux tube (the distance between charges) reaches its minimum in the limit when there is only a "magnetic" field ($E_{KK} = 0$).

III. DISCUSSION

As the relative strengths of the Kaluza-Klein fields are varied we find that the solutions to the metric in Eq. (1) evolve in a very interesting and suggestive way :

1. $0 \le H_{KK} < E_{KK}$. The solution is ***a WH-like object*** located between two surfaces at $\pm r_0$ where the reduction of 5D to 4D spacetime breaks down. The cross-sectional size of these solution increases as r goes from 0 to $\pm r_0$. The throat between the $\pm r_0$ surfaces is filled with "electric"and/or "magnetic" flux. As the strength of the "magnetic" field is increased the longitudinal distance between the $\pm r_0$ surfaces increases, and the cross-sectional size does not increase as rapidly as $r \to r_0$.

2. $H_{KK} = E_{KK}$. In this case the solution is ***an infinite flux tube*** filled with constant "electrical" and "magnetic" fields, and with the charges disposed at $\pm\infty$. The cross-sectional size of this solution is constant ($a = const.$). Essentially, as the magnetic field strength is increased one can think that the previous solutions are stretched so that the $\pm r_0$ surfaces are taken to $\pm\infty$ and the cross section becomes constant.

3. $0 \le E_{KK} < H_{KK}$. In this case we have ***a finite flux tube*** located between two (+) and (-) "magnetic" and/or "electric" charges located at $\pm r_0$. Thus the longitudinal size of this object is again finite, but now the cross sectional size decreases as $r \to r_0$. At $r = \pm r_0$ this solution has real singularities which we interpret as the locations of the charges. This solution is very similar to the confinement mechanism in QCD where two quarks are disposed at the ends of a flux tube with color electrical and magnetic fields running between the quarks. In the $E_{KK} = 0$ limit we find two opposite "magnetic" charges confined to a spacetime of fixed volume. This may indicate why single, asymptotic magnetic charges have never been observed in Nature : they are permanently confined into monopole-antimonopole pairs of some fixed volume. Finally, we note that in Ref. [17] some similar mappings between 4D gravity and non-Abelian theory are discussed.

The evolution of the solution from a WH-like object, to an infinite flux tube, to a finite flux tube, as the relative strengths of the fields is varied, is summarized in Fig.1. This allows two complimentary conclusions : First, if one takes some Wheeler-like model of electric charge as in Ref. [12] then it can be seen that if the magnetic field becomes too strong the WH-like solution is destroyed and with it the Wheeler-like model of electric charge. Second, if one concentrates a sufficently strong electric field (*i.e.* $E_{KK} > H_{KK}$) into some small region of spacetime one is led to the science fiction-like possibility that one may be able ***to "open" the finite flux tubes into a WH-like configuration***. This conjecture assumes some kind of spacetime foam model where the vacuum is populated by virtual flux tubes filled with virtual "magnetic" and/or "electric" fields.

Starting from the solutions obtained here we see that in 5D gravity there is a distinction between "electrical" and "magnetic" Kaluza-Klein fields. This can be contrasted with the 4D electrogravity Reissner-Nordström solution which is the same for the electrical and magnetic charges.

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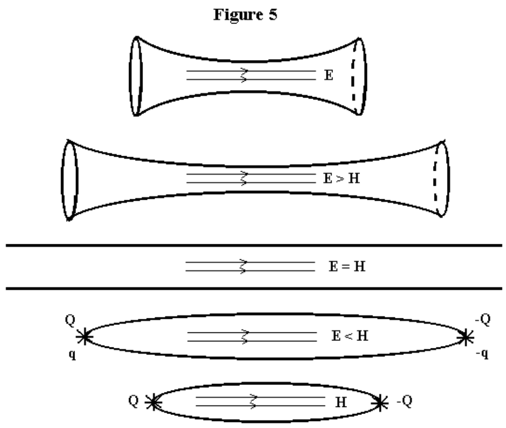

FIG. 1. The evolution of the "electric"/"magnetic" solutions as a function of the relative strengths between the two charges

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